Mathematical Social Sciences 21 (1991) 67–79
North-Holland

COMPLETELY FEASIBLE AND CONTINUOUS IMPLEMENTATION OF THE LINDAHL CORRESPONDENCE WITH ANY NUMBER OF GOODS

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Communicated by M. Kaneko
Received 10 October 1990
Revised 30 October 1990

This paper deals mainly with the problem of designing mechanisms whose Nash allocations coincide with the Lindahl allocations for public goods economies with any number of private and public goods. The mechanism presented here improves the previous mechanisms by introducing two new features. One is that the mechanism is balanced (not merely weakly balanced). The other is that the level of public goods is provided with the marginal cost pricing rule for both equilibrium and disequilibrium messages so that the single-valued input demand outcome function is obtained by the Shephard lemma and the prices of public goods equal the minimum-unit-cost functions. Besides, the mechanism is single-valued, individually feasible, and continuous.

Key words: Lindahl allocations; completely feasible and continuous implementation; any number of goods.

1. Introduction

Since the seminal paper of Hurwicz (1972) on mechanism design theory, there have been many mechanisms that solve free-rider problems in the sense that they result in Pareto efficient allocations for public goods economies at Nash equilibria. Groves and Ledyard (1977) were the first to propose a mechanism that yields Pareto optimal Nash allocations even though their mechanism is neither individually rational nor individually feasible (i.e. individuals can be worse off at the equilibria than at their initial holdings and it may yield an allocation that is not in the consumption set for some agent). Subsequently, Hurwicz (1979) and Walker (1981) gave mechanisms with balanced and smooth outcome functions whose Nash allocations are precisely Lindahl allocations which are individually rational and Pareto

* We wish to thank Professor M. Kaneko and two anonymous referees for helpful comments and suggestions. Of course, any remaining errors are our own.

efficient.[1] Their mechanisms, however, are still not individually feasible. Furthermore, for economies with more than one private good, Walker's mechanism (also the mechanism of Groves and Ledyard, 1977) is not really a game form but a combination of a game and the market mechanism. Such a mixed mechanism is not single-valued and requires the presence of an auctioneer, and has as well an incentive-compatibility problem since the market mechanism has these problems and agents may not be 'price-takers' for the small number of agents. For a detail discussion about these issues, see Tian (1989).

Thus, all of the mechanisms mentioned above are not individually feasible. In fact, there does not exist any completely feasible (i.e. individually feasible and balanced) mechanism that implements the Lindahl correspondence because the correspondence violates Maskin's (1977) monotonicity condition when boundary equilibrium allocations occur. Nevertheless, it is still possible to design a completely feasible mechanism whose Nash allocations coincide with a slightly larger set than the set of Lindahl allocations, namely, the set of constrained Lindahl allocations.[2] Hurwicz, Maskin and Postlewaite (1984) presented a mechanism implementing the constrained Lindahl correspondence. Their mechanism is individually feasible and balanced but discontinuous; small variations in an agent's strategy choice may lead to large jumps in the resulting allocations. Thus, the aforementioned mechanisms do not guarantee complete feasibility and/or continuity. Recently, Tian (1990a) gave a completely feasible and continuous mechanism which implements the Lindahl correspondence for economies with one private and one public good. For public goods economies with any number of goods, Tian (1989) proposed a feasible and continuous mechanism which implements the Lindahl correspondence. But this mechanism is not balanced and public goods are produced in a technologically inefficient way out of equilibria.

Similar to motivations behind designing feasible and continuous mechanisms, the balancedness condition and the efficiency of production become important when implementation requires that disequilibrium points be used to compute allocations.[3] These properties are highly desirable if we seriously consider accepting the mechanism. Indeed, in the cases of small errors in communication, fallibility of information transmission, or a given stopping rule, one often has to use disequilibrium points to compute allocations. If a mechanism is merely weakly balanced, the allocation at a disequilibrium point may be less than the total endowments. Should this happens, some resources are not completely used (i.e. free disposal occurs). However, when preferences of agents are strictly monotone, individuals will be bet-

[1] Walker's mechanism has the advantage of using a minimal-sized message space.

[2] A constrained Lindahl allocation differs from an ordinary Lindahl allocation only in the way that each agent maximizes his preferences not only subject to his budget constraint but also subject to total endowments available to the economy (see Hurwicz, 1986, or Tian, 1988).

[3] This is a common phenomenon, say, for the market mechanism; we never fully attain the equilibrium because technologies of production and preferences of agents change over time.

ter off if the unused resources are distributed to them.[4] Also, if production of producing public goods is not efficient, the cost of producing public goods is larger than the minimum cost. In both cases the mechanism results in Pareto inefficient allocations since the balancedness of allocations and the minimum cost of production are necessary conditions for Pareto efficiency. Thus, under the monotonicity of preferences, a weakly balanced allocation is dominated by a balanced allocation and a productively inefficient allocation is dominated by a productively efficient allocation. On the other hand, if a mechanism is balanced and public goods are produced at the lowest possible cost, even if disequilibrium points are used to compute the allocations, one can guarantee the efficiency of production and that resources are not wasted or destroyed.[5]

This paper presents a mechanism whose Nash allocations coincide with the Lindahl allocations for public goods economies with any number of private and public goods. We do so by assuming that the technology of producing public goods exhibits constant returns to scale and can be represented by a strictly quasiconcave production function. We further assume that preferences of agents satisfy the monotonicity and convexity conditions, and in addition, any private-goods interior allocation is strictly preferred to any private-goods boundary allocation. As will be noted, this mechanism improves the mechanisms in the existing literature by introducing two new features. One is that the mechanism is balanced. The other is that the level of public goods is provided with the minimal cost pricing rule for both equilibrium and disequilibrium messages so that the single-valued input demand outcome function is obtained by the Shephard lemma and the prices of public goods are equal to the minimum-unit-cost functions. Besides, this mechanism is single-valued, individually feasible, and continuous, and furthermore, has a message space of finite dimension.

The plan of this paper is as follows. Section 2 sets forth a public goods model and presents a mechanism that has the desirable properties mentioned above. Section 3 shows that this mechanism fully implements the Lindahl correspondence.

2. Public goods model and mechanism

2.1. Economic environments

In a model of a public goods economy, there are n agents who consume L private goods and K public goods, x being private and y public. Throughout this paper, the

[4] We thank Professor L. Hurwicz for pointing out these observations to us which stimulated our interest in designing balanced mechanisms.

[5] Note that these disequilibrium allocations do not necessarily result in Pareto inefficient allocations since the set of Lindahl allocations, which coincides with the set of Nash allocations of the mechanism that implements the Lindahl correspondence, is, in general, much smaller than the set of Pareto efficient allocations (the set of all points on the contract curve). Thus a disequilibrium allocation can still be a Pareto efficient allocation.

subscript is used to index agents while the superscript is used to index goods. Denote by $N=\{1,2,\ldots,n\}$ the set of agents. Each agent's characteristic is denoted by $e_i=(w_i,R_i)$, where w_i is the initial endowment vector of private goods and R_i the preference ordering (P_i denotes the asymmetric part of the preference R_i). We assume that there are no initial endowments of public goods, but that the public goods can be produced from the private goods by a production function $f\colon \mathbb{R}_+^L \to \mathbb{R}_+^K$. An economy is the full vector $e=(e_1,\ldots,e_n,f(\cdot))$ and the set of all such economies is denoted by E. The following assumptions are made on E:

Assumption 1. $n \geqq 3$.[6]

Assumption 2. $w_i \geq 0$ *for all* $i\in N$ *and* $w>0$. *Here* $w=\sum_{i=1}^n w_i$.

Assumption 3. *The preference R_i is reflexive, transitive, total, convex,*[7] *and strictly monotone increasing on* $\mathbb{R}_+^{L+K}$.

Assumption 4. *For all* $i\in N$, $(x_i,y)P_i(x_i',y')$ *for all* $x_i\in\mathbb{R}_{++}^L$, $x_i'\in\partial\mathbb{R}_+^L$, *and* $y,y'\in\mathbb{R}_+^K$, *where* $\partial\mathbb{R}_+^m$ *is the boundary of* $\mathbb{R}_+^m$.[8]

Assumption 5. *The production function $f(v)$ is strictly quasi-concave, homogeneous of degree one, and increasing in v.*[9]

Remark 1. Assumption 4 was called 'indispensability of money' by Mas-Colell (1980) when there is only one private good which is considered as money. This assumption cannot be dispensed with in Theorem 1 below. Tian (1987, 1988) showed that the (constrained) Lindahl correspondence violates Maskin's (1977) monotonicity condition only under Assumptions 1-3 and thus cannot be Nash-implemented by a completely feasible mechanism.

2.2. Lindahl allocations

An allocation $(x,y)=(x_1,\ldots,x_n,y)$ is *feasible* for an economy e if $(x,y)\in\mathbb{R}_+^{nL+K}$, $y=f(v)$, and $\sum_{i=1}^n x_i+v\leqq\sum_{i=1}^n w_i$.

An allocation (x^*,y^*) is a *Lindahl allocation* for an economy e if it is feasible and there is a price vector $p^*\in\mathbb{R}_+^L$, personalized price vectors $q_i^*\in\mathbb{R}^K$, one for each i,

[6] As usual, vector inequalities are defined as follows: Let $a,b\in\mathbb{R}^m$. Then $a\geqq b$ means $a_s\geq b_s$ for all $s=1,\ldots,m$; $a\geq b$ means $a\geqq b$ but $a\neq b$; $a>b$ means $a_s>b_s$ for all $s=1,\ldots,m$.

[7] P_i is convex if, for bundles a,b,c and $0<\lambda\leqq 1$, $c=\lambda a+(1-\lambda)b$, the relation $a\,P_i\,b$ implies $c\,P_i\,b$ on $\mathbb{R}_+^{L+K}$.

[8] If Assumption 4 is replaced by Assumption 4': for all $i\in N$, $(x_i,y)P_i(x_i',y')$ for all $(x_i,y)\in\mathbb{R}_{++}^{L+K}$, $(x_i',y')\in\partial\mathbb{R}_+^{L+K}$, the mechanism in the paper can be slightly modified so that the dimension of the message space can be reduced by nK dimensions. (cf. Tian, 1990a).

[9] These conditions on f can be weakened to the necessary and sufficient conditions for differentiability of cost functions which were given by Färe and Primont (1986).

(1) $p^* \cdot x_i^* + q_i^* \cdot y^* \leqq p^* \cdot w_i$ for all $i \in N$;
(2) for all $i \in N$, $(x_i, y) P_i (x_i^*, y^*)$ implies $p^* \cdot x_i + q_i^* \cdot y > p^* \cdot w_i$; and
(3) $q^* \cdot y^* - p^* \cdot v^* = 0$,
where $v^* = \sum_{i=1}^n w_i - \sum_{i=1}^n x_i$ and $\sum_{i=1}^n q_i^* = q^*$. Note that condition (3) is a familiar zero-profit condition under constant returns to scale. Denote by $L(e)$ the set of all such allocations.

2.3. Mechanism

Let M_i denote the ith message domain. Its elements are written as m_i and called the messages. Let $M = \prod_{i=1}^n M_i$ denote the message space. Let $h: M \to Z$ denote the outcome function, or more explicitly, $h_i(m) = (X_i(m), Y(m), \hat{v}(m))$, where $X_i(m)$ is the ith agent's outcome function for private goods, $Y(m)$ the outcome function for public goods, and $\hat{v}(m)$ is the outcome function for inputs used to produce $Y(m)$ units of public goods. A mechanism consists of $\langle M, h \rangle$ defined on E.

A message $m^* = (m_1^*, \ldots, m_n^*) \in M$ is said to be a *Nash equilibrium* (NE) of the mechanism $\langle M, h \rangle$ for an economy e if for any $i \in N$ and for all $m_i \in M_i$,

$$(X_i(m^*), Y(m^*)) R_i (X_i(m^*/m_i, i), Y(m^*/m_i, i)), \tag{1}$$

where $(m^*/m_i, i) = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_n^*)$. The $h(m^*)$ is then called a *Nash (equilibrium) allocation*. Denote by $V_{M,h}(e)$ the set of all such Nash equilibria and by $N_{M,h}(e)$ the set of all such Nash (equilibrium) allocations. A mechanism $\langle M, h \rangle$ is said to fully *Nash-implement* the Lindahl correspondence L on E, if, for all $e \in E$, $N_{M,h}(e) = L(e)$. A mechanism $\langle M, h \rangle$ is said to be *individually feasible* if $(X(m), Y(m)) \in \mathbb{R}_+^{nL+K}$ for all $m \in M$. A mechanism $\langle M, h \rangle$ is said to be *weakly balanced* if for all $m \in M$,

$$\sum_{j=1}^{N} X_j(m) + \hat{v}(m) \leqq \sum_{j=1}^{n} w_j. \tag{2}$$

A mechanism $\langle M, h \rangle$ is said to be *balanced* if the above equation (2) holds with equality for all $m \in M$. A mechanism $\langle M, h \rangle$ is said to be *completely feasible* (*feasible*) if it is individually feasible and balanced (weakly balanced). Sometimes we say that an outcome function is individually feasible, balanced, or continuous if the mechanism is individually feasible, balanced, or continuous.

In what follows we construct a single-valued, completely feasible, and continuous mechanism. Before the formal presentation of this mechanism, we briefly describe the mechanism as follows. The designer first determines the prices of private goods (given in (4)). To make the outcome function for inputs single-valued and the zero-profit condition hold, the prices of public goods are determined by the minimum-uni-cost functions from which the input demand function is obtained by using Shephard's lemma. And the personalized prices for public goods are defined according to agent's announcement about these prices (given in (7) below). Then define a feasible choice correspondence B_y (given in (8) below) for public goods that can be produced with the total endowments and can be purchased by all agents. The outcome $Y(m)$ for public goods will be chosen from $B_y(m)$ so that it is the closest to the sum of the contributions for public goods that agents are willing to pay (see (9) below). That is, the designer tries to satisfy the agents' desires as best as he can. After that, define a completely feasible choice correspondence B_x (defined below) for private goods so that for all $m \in M$ allocations of private and public goods are completely feasible and satisfy the budget constraints of all agents with equality. The outcome for private goods $X(m)$ will be chosen from the completely feasible set $B_x(m)$ so that it is the closest to the sum of the contributions that each agent is willing to pay. It will be seen that allocation $(X(m), Y(m))$ resulting from the message m is completely feasible, continuous, and single-valued for all $m \in M$.

Now we turn to the formal construction of the mechanism. For each $i \in N$, it is assumed that his message domain is of the form

$$M_i = \mathbb{R}^L_{++} \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^{nL}_{+}. \tag{3}$$

A generic element of M_i is $(p_i, \phi_i, y_i, (x_{i1}, \ldots, x_{in}))$, where p_i denotes the price vector of private goods proposed by agent i; ϕ_i denotes the price vector of public goods proposed by agent i for use in other agents' budget constraints; y_i denotes the proposed level of public goods that agent i is willing to contribute (a negative y_i means the agent wants to receive a subsidy from the society); and x_{ij} denotes the contribution of private goods that agent i is willing to make to agent j (a negative x_{ij} means agent i wants to get $-x_{ij}$ amount of private goods from agent j).

Define the price vector for the private goods $p: M \to \mathbb{R}^L_{++}$ by

$$p(m) = \begin{cases} \sum_{i=1}^{n} \frac{a_i}{a} p_i, & \text{if } a > 0, \\ \sum_{i=1}^{n} \frac{1}{n} p_i, & \text{if } a = 0, \end{cases} \tag{4}$$

which is continuous. Here $a_i = \sum_{j,s \neq i}^{n} \|p_j - p_s\|$, $a = \sum_{i=1}^{n} a_i$, and $\|\cdot\|$ is the Euclidian norm. Notice that even though $p(m)$ is a function of the component $(p_1, \ldots, p_n)$ of the message m only, we can write it as the function of m without loss of generality.

Let the prices $q^k(m)$ of public goods be equal to the minimum-unit-cost functions which are dual to the production functions $f^k (k = 1, \ldots, K)$ (cf. Samuelson, 1953, and Shephard, 1953) and defined by

$$c^k(p(m)) = \min\{p(m) \cdot v : f^k(v) = 1\}, \tag{5}$$

which are differentiable, homogeneous of degree one, and concave in $p(m)$. Thus $q^k(m) = c^k(p(m))$ for $k = 1, \ldots, K$.

Then, by Shephard's lemma, the input demand functions $v^k(y, p(m))$ are given by

$$v^k(y, p(m)) = y^k \frac{\partial c^k(p(m))}{\partial p(m)}, \quad k = 1, \ldots, K,$$

and the aggregate input demand function is

$$v(y, p(m)) = \sum_{k=1}^{K} v^k(y, p(m)) = y \cdot Dc(p(m)),$$

where

$$Dc(p(m)) = \left(\frac{\partial c^1(p(m))}{\partial p(m)}, \ldots, \frac{\partial c^K(p(m))}{\partial p(m)}\right).$$

Thus, by the homogeneity of degree one of $c(p(m))$, we have

$$q(m) \cdot y - p(m) \cdot v(y, p(m)) = 0, \tag{6}$$

which means the zero-profit condition holds for all $m \in M$.

Define the personalized price of each public good k for the ith consumer by

$$q_i^k(m) = b_i^k(m) + \sum_{j=1}^{n} a_{ij}^k \phi_j^k, \tag{7}$$

where $\sum_{i=1}^{n} b_i^k(m) = q^k(m)$, $\sum_{i=1}^{n} a_{ij}^k = 0$, $a_{ii}^k = 0$, and $\sum_{j=1}^{n} |a_{ij}^k| > 0$, for $i \in N$ and $k = 1$ to K. In addition, a_{ij}^k are chosen so that any personalized price vector q_i can be attained from the equation (7) by suitably choosing $(\phi_1, \ldots, \phi_n)$.[10] Observe that, by construction, $\sum_{i=1}^{n} q_i(m) = q(m)$ for all $m \in M$ and each agent's personalized prices are independent of his own message (i.e. $q_i(m^*) = q_i(m^*/m_i, i)$, for any $m_i \in M_i$). Here $q_i(m) = (q_i^1(m), \ldots, q_i^K(m))$ and $q(m) = (q^1(m), \ldots, q^K(m))$.

Define the correspondence B_y: $M \to 2^{\mathbb{R}^K}$ by

$$B_y(m) = \{y \in \mathbb{R}_+^K : p(m) \cdot w_i - q_i(m) \cdot y \geqq 0 \ \forall i \in N \ \& \ w - v(y, p(m)) \geqq 0\}, \tag{8}$$

which is clearly non-empty, compact, convex, and continuous on $m \in M$.

Define the outcome function for public goods $Y: M \to B_y$ by

$$Y(m) = \left\{y : \min_{y \in B_y(m)} \|y - \tilde{y}\|\right\}, \tag{9}$$

which is the closest to $\tilde{y}$. Here $\tilde{y} = \sum_{i=1}^{n} y_i$. Then $Y(m)$ is single-valued and continuous on M.[11]

For each individual i, define the taxing function $T_i: M \to \mathbb{R}$ by

$$T_i(m) = q_i(m) \cdot Y(m). \tag{10}$$

Then

$$\sum_{i=1}^{n} T_i(m) = q(m) \cdot Y(m). \tag{11}$$

[10] Note that personalized prices propesed by Hurwicz (1979) and Walker (1981) are special cases of ours.

[11] This is because $Y(m)$ is an upper semi-continuous correspondence by Berge's Maximum Theorem (see Debreu, 1959, p. 19) and single-valued (see Mas-Colell, 1985, p. 28).

To determine the level of private goods for each individual i, define a completely feasible correspondence $B_x: M \to 2^{\mathbb{R}^{nl}_+}$ by

$$B_x(m) = \left\{x \in \mathbb{R}^{nL}_+ : \sum_{i=1}^{n} x_i + \hat{v}(m) = w \,\&\, p(m)\cdot x_i + T_i(m) = p(m)\cdot w_i \ \forall i \in N\right\}, \tag{12}$$

where $\hat{v}(m) = v(Y(m), p(m)) = Y(m)\cdot Dc(p(m))$ is the input demand outcome function. B_x is clearly compact and convex. We will show it is also non-empty and continuous in the appendix.

Let $\tilde{x}_j = \sum_{i=1}^{n} x_{ij}$, which is the sum of contributions that agents are willing to make to agent j and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$.

The outcome function $X(m): M \to B_x$ is given by

$$X(m) = \left\{z \in \mathbb{R}^{nL}_+ : \min_{z \in B_x(m)} \|z - \tilde{x}\|\right\}, \tag{13}$$

which is the closest to $\tilde{x}$. Then $X(m)$ is single-valued and continuous on M. Also, since $(X(m), Y(m)) \in \mathbb{R}^{nL+K}_+$ and

$$\sum_{i=1}^{n} X_i(m) + \hat{v}(m) = \sum_{i=1}^{n} w_i, \tag{14}$$

for all $m \in M$, the outcome function specified above is single-valued, completely feasible, and continuous on M.

Remark 2. Observe that in the case of $L=1$ the part for the private goods game is not needed. Also, if there are no public goods (i.e. $K=0$), public goods economies reduce to private goods economies and the mechanism reduces to a mechanism proposed in Tian (1990b) which Nash implements the (constrained) Walrasian correspondence.

3. Main results

The remainder of this paper is devoted to the proof of equivalence between Nash allocations and Lindahl allocations. Lemma 1 in the following is a preliminary result used to prove that every Nash allocation is a Lindahl allocation, which is stated in Theorem 1. Theorem 2 proves that every Lindahl allocation is a Nash allocation.

Lemma 1. *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $X_i(m^*) \in \mathbb{R}^L_{++}$ for all $i \in N$.*

Proof. Suppose, by way of contradiction, that $X_i(m^*) \in \partial\mathbb{R}^L_+$ for some agent, say, agent 1. Since $p(m^*) > 0$, there is some $\tilde{x}_1 \in \mathbb{R}^L_{++}$ such that $p(m^*)\cdot\tilde{x}_1 = p(m^*)\cdot w_1$, $\tilde{x}_1 \leq w$, and $(\tilde{x}_1, 0)\, P_1\, (X_1(m^*), Y(m^*))$ by Assumption 4. Define $\tilde{x}_2, \ldots, \tilde{x}_i, \ldots, \tilde{x}_n$ as follows:

$$\tilde{x}_2 = \frac{p(m^*) \cdot w_2}{\sum_{j=2}^{n} p(m^*) \cdot w_j} [w - \tilde{x}_1], \tag{15}$$

$$\tilde{x}_i = \frac{p(m^*) \cdot w_i}{\sum_{j=i}^{n} p(m^*) \cdot w_j} \left[w - \sum_{j=1}^{i-1} \tilde{x}_j \right], \tag{16}$$

$$\tilde{x}_n = w - \sum_{j=1}^{n-1} \tilde{x}_j. \tag{17}$$

Since $\tilde{x}_1 \leq \sum_{j=1}^{n} w_j$, it can be easily verified, by reduction, that

$$\sum_{j=1}^{n} w_j - \sum_{j=1}^{i-1} \tilde{x}_j \geq 0,$$

for $i=2,\dots,n$, and thus $\tilde{x}_i \in \mathbb{R}_+^L$ and

$$p(m^*) \cdot \tilde{x}_i = p(m^*) \cdot w_i,$$

for all $i=2,\dots,n$. Also, from (17) we have

$$\sum_{j=1}^{n} \tilde{x}_j = \sum_{j=1}^{n} w_j.$$

Now suppose that agent 1 chooses $y_1 = -\sum_{j=2}^{n} y_j^*$, $x_{1j} = \tilde{x}_j - \sum_{k=2}^{n} x_{kj}^*$, for all $j \in N$, and keeps other messages unchanged. Then, $0 \in B_y(m^*/m_1, 1)$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \in B_x(m^*/m_1, 1)$, $y_1 + \sum_{k=2}^{n} y_k^* = 0$, and $\tilde{x}_j = x_{1j} + \sum_{k=2}^{n} x_{kj}^*$, for all $j \in N$. Hence, we have $Y(m^*/m_1, 1) = 0$ and $X_j(m^*/m_1, 1) = \tilde{x}_j$ and thus $(X_1(m^*/m_1, 1), 0) P_1 (X_1(m^*), Y(m^*))$. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

We now turn to prove the main results of this paper in the following theorems.

Theorem 1. *Under Assumptions 1–5, if the mechanism defined above has a Nash equilibrium m^*, then $(X(m^*), Y(m^*))$ is a Lindahl allocation with the price system $(p(m^*), q_1(m^*), \dots, q_n(m^*))$, i.e. $N_{M,h}(e) \subseteq L(e)$.*

Proof. Let m^* be a Nash equilibrium. We prove that $(X(m^*), Y(m^*))$ is a Lindahl allocation with the price vector system $(p(m^*), q_1(m^*), \dots, q_n(m^*)) \in \mathbb{R}_+^{L+nK}$. By the construction of the mechanism, we know $p(m^*) \cdot X_i(m^*) + q_i(m^*) \cdot Y(m^*) = p(m^*) \cdot w_i$ and the zero-profit condition holds. Also, by the definition of the mechanism, $(X(m^*), Y(m^*))$ is feasible. So we only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that there exists some agent, say, agent 1 and $(x_1, y) \in \mathbb{R}_+^{L+K}$ such that $(x_1, y) P_1 (X_1(m^*), Y(m^*))$ and $p(m^*) \cdot x_1 + q_1(m^*) \cdot y \leqq p(m^*) \cdot w_1$. Because of strict monotonicity of preferences, it will be enough to confine ourselves to the case of $p(m^*) \cdot x_1 + q_1(m^*) \cdot y = p(m^*) \cdot w_1$. Let $(x_{1\lambda}, y_\lambda, v_\lambda) = (\lambda x_1 + (1-\lambda) X_1(m^*),\ \lambda y + (1-\lambda) Y(m^*),\ \lambda v(y, p(m^*)) + (1-\lambda) \cdot \hat{v}(m^*))$. Then, by convexity of preferences and constant returns to scale, we have $(x_{1\lambda}, y_\lambda) P_1 (X_1(m^*), Y(m^*))$ and $v_\lambda = v_\lambda(y_\lambda, p(m^*))$, for all $0 < \lambda < 1$. Also $(x_{1\lambda}, y_\lambda) \in \mathbb{R}_+^{L+K}$ and $p(m^*) \cdot x_{1\lambda} + q_1(m^*) \cdot y_\lambda = p(m^*) \cdot w_1$.

Since $p(m^*)\in \mathbb{R}^{nL}_{++}$ and $X(m^*)\in \mathbb{R}^{nL}_{++}$, we must have $p(m^*)\cdot w_j - q_j(m^*)\cdot Y(m^*)>0$, for all $j\in N$, and $w=\hat{v}(m^*)+\sum_{j=1}^{n} X_j(m^*)>\hat{v}(m^*)+X_1(m^*)$. Then, we have $p(m^*)\cdot w_j - q_j(m^*)\cdot y_\lambda>0$, for all $j\in N$, and $x_{1\lambda}+v_\lambda<w$ as λ is a sufficiently small positive number.

Define $x_{2\lambda},\dots,x_{i\lambda},\dots,x_{n\lambda}$ as follows:

$$x_{2\lambda}=\frac{p(m^*)\cdot w_2-q_2(m^*)\cdot y_\lambda}{p(m)\cdot[w-v_\lambda-x_{1\lambda}]}[w-v_\lambda-x_{1\lambda}], \tag{18}$$

$$x_{i\lambda}=\frac{p(m^*)\cdot w_i-q_i(m^*)\cdot y_\lambda}{p(m^*)\cdot[w-v_\lambda-\sum_{j=1}^{i-1}x_{j\lambda}]}\left[w-v_\lambda-\sum_{j=1}^{i-1}x_{j\lambda}\right], \tag{19}$$

$$x_{n\lambda}=\sum_{j=1}^{n}w_j-v_\lambda-\sum_{j=1}^{n-1}x_{j\lambda}. \tag{20}$$

Then

$$\sum_{j=1}^{n}x_{j\lambda}+v_\lambda=\sum_{j=1}^{n}w_j,$$

and

$$p(m^*)\cdot x_{i\lambda}+q_i(m^*)\cdot y_\lambda=p(m^*)\cdot w_i,$$

for all $i\in N$. Also, since $x_{1\lambda}+v_\lambda<w$ and

$$\frac{p(m^*)\cdot w_i-q_i(m^*)\cdot y_\lambda}{p(m^*)\cdot[w-v_\lambda(m)-\sum_{j=1}^{i-1}x_{j\lambda}]}=\frac{p(m^*)\cdot w_i-q_i(m^*)\cdot y_\lambda}{\sum_{j=i}^{n}p(m^*)\cdot w_j-q_j(m^*)\cdot y_\lambda}<1,$$

for $i=2,\dots,n$, we can easily see that $x_{2\lambda},\dots,x_{i\lambda},\dots,x_{n\lambda}$ are non-negative.

Now suppose that player 1 chooses $y_1=y_\lambda-\sum_{j\neq 1}^{n}y_j^*$ and $x_{1j}=x_{j\lambda}-\sum_{k=2}^{n}x_{kj}^*$, for all $j\in N$, and keeps other messages unchanged. Then $y_\lambda\in B_y(m^*/m_1,1)$ and $x_\lambda\in B_x(m^*/m_1,1)$. Therefore, we have $Y(m^*/m_1,1)=y_\lambda$, $X_1(m^*/m_1,1)=x_{1\lambda}$, and $\hat{v}(m^*/m_1,1)=v_\lambda$. Hence $(X_1(m^*/m_1,1),Y(m^*/m_1,1))P_1(X_1(m^*),Y(m^*))$. This contradicts $(X(m^*),Y(m^*))\in N_{M,h}(e)$. So $(X(m^*),Y(m^*))$ is a Lindahl allocation. Q.E.D.

Theorem 2. *Under Assumptions* 1-5, *if* (x^*,y^*) *is a Lindahl allocation with the price system* $(p^*,q_1^*,\dots,q_n^*)$, *then there is a Nash equilibrium* m^* *of the mechanism such that* $Y(m^*)=y^*$, $X_i(m^*)=x_i^*$, $q_i(m^*)=q_i^*$, *for all* $i\in N$, $p(m^*)=p^*$, *i.e.* $L(e)\subseteq N_{M,h}(e)$.

Proof. We need to show that there is a message m^* such that (x^*,y^*) is a Nash allocation. Let $p_i^*=p^*$, and let $(y_1^*,\dots,y_n^*,x_1^*,\dots,x_n^*,\phi_1^*,\dots,\phi_n^*)$ be the solution of the following linear equations system:

$$\sum_{i=1}^{n}y_i=y^*,$$

$$\sum_{i=1}^{n}x_i=x^*, \tag{21}$$

$$q_i^k=b_i^k(m^*)+\sum_{j=1}^{n}a_{ij}^k\phi_{kj},$$

for $k=1,\dots,K$. Then, $p(m^*)=p^*$, $q(m^*)=q^*$, $X_i(m^*)=x_i^*$, $Y(m^*)=y^*$, and $q_i(m^*)=q_i^*$ for all $i\in N$. Notice that $(p(m^*/m_i,i), q_i(m^*/m_i,i))=(p(m^*),q_i(m^*))$, for all $m_i\in M_i$, $(X(m^*/m_i,i), Y(m^*/m_i,i))\in \mathbb{R}_+^{L+K}$, and $p(m^*)\cdot X_i(m^*/m_i,i)+q_i(m^*)Y(m^*/m_i,i)=p(m^*)\cdot w_i$, for all $i\in N$ and $m_i\in M_i$. From $(x^*,y^*)\in L(e)$, we have $(X_i(m^*),Y(m^*))\,R_i\,(X_i(m^*/m_i,i),Y(m^*/m_i,i))$ for all $m_i\in M_i$ and thus $(X(m^*),Y(m^*))$ is a Nash equilibrium allocation. Q.E.D.

Thus, from the above discussions we can conclude that for public goods economies E there exists a single-valued, completely feasible, and continuous mechanism which fully Nash-implements the Lindahl correspondence.

When $K=0$, public goods economies reduce to private goods economies and the mechanism presented in this paper reduces to a mechanism in Tian (1990b). Thus we have the following corollary.

Corollary 1. *For private goods economies satisfying Assumptions* 1-3,[12] *there exists a single-valued, completely feasible, and continuous mechanism which fully Nash-implements the constrained Walrasian correspondence.*

It may be remarked that our mechanism is almost everywhere differentiable on the message space. Furthermore, if Assumption 4 is replaced by Assumption 4′ in footnote 8, the outcome function becomes differentiable on some neighborhood of every Nash equilibrium so that one can use the differential approach to find Nash equilibrium (cf. Li, Nakamura and Tian, 1990). Similar to those mechanisms in Hurwicz, Maskin and Postlewaite (1984), Postlewaite and Wettstein (1989), and Tian (1989), the mechanism can also be easily extended to the case where the initial endowments are private information. This situation would certainly increase the size of the message space but would reduce the information requirements on the designer.

Appendix

Proof of non-emptyness of $B_x(m)$. Define $x_1(m),\dots,x_i(m),\dots,x_n(m)$ as follows:

$$x_1(m)=\begin{cases}\dfrac{p(m)\cdot w_1-q_1(m)\cdot Y(m)}{p(m)\cdot[w-\hat{v}(m)]}[w-\hat{v}(m)], & \text{if } w-\hat{v}(m)\ge 0,\\ 0, & \text{otherwise;}\end{cases} \tag{22}$$

[12] In fact, Tian (1990b) used a much weaker assumption than Assumption 3 about preferences, namely only monotonicity of preferences is assumed so that preferences may be non-total, non-transitive and non-convex.

$$x_i(m) = \begin{cases} \dfrac{p(m)\cdot w_i - q_i(m)\cdot Y(m)}{p(m)\cdot \Delta_i(m)}[\Delta_i(m)], & \text{if } \Delta_i(m) \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (i=2,\dots,n-1); \tag{23}$$

$$x_n(m) = \sum_{j=1}^{n} w_j - \hat{v}(m) - \sum_{j=1}^{n-1} x_j(m), \tag{24}$$

where $\Delta_i(m) = w - \hat{v}(m) - \sum_{j=1}^{i-1} x_j(m)$. Since $x_1(m) \leqq \sum_{j=1}^{n} w_j - \hat{v}(m)$ and $p(m)\cdot w_i - q_i(m)\cdot Y(m) \geqq 0$, for all i, it can be easily verified, by reduction, that

$$\sum_{j=1}^{n} w_j - \sum_{j=1}^{i-1} x_j(m) \geqq 0,$$

for $i=2,\dots,n$. Thus $x_j(m)\in \mathbb{R}_+^L$ for all $j\in N$ and, from (24),

$$\sum_{j=1}^{n} x_j(m) + \hat{v}(m) = \sum_{j=1}^{n} w_j. \tag{25}$$

Also, from the constructions of $x_i(m)$, we have

$$p(m^*)\cdot x_i(m) + q_i(m)\cdot Y(m) \leqq p(m^*)\cdot w_i. \tag{26}$$

Now, from the zero-profit equation (6) and the balanced condition (25), the above equation (26) must hold with equality for all $i\in N$. Hence $x(m)\in B_x(m)$. So $B_x(m)$ is non-empty for all $m\in M$. Q.E.D.

Proof of continuity of B_x. $B_x(m)$ is clearly upper semi-continuous for all $m\in M$. We only need to show $B_x(m)$ is also lower semi-continuous at m. In fact, for the function $x(m)$ defined by (22)–(24), we have $x(m)\in B_x(m)$ for all $m\in M$. Thus, for any sequence $\{m_t\}$, with $m_t\to m$, $x(m_t)\in B_x(m_t)$. Now, if $x_i(m)\geq 0$, it is clear that $x(m_t)\to x(m)$ as $m_t\to m$. If $x_i(m)=0$, we also have $0\leqq x_i(m_t)\leqq \Delta(m_t)\to \Delta_i(m)=0$ as $m_t\to m$. Thus, in both cases we have $x(m_t)\to x(m)$ as $m_t\to m$. So $B_x(m)$ is lower semi-continuous for all $m\in M$. Q.E.D.

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