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# Ratio–Lindahl and Ratio Equilibria with Many Goods*

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This paper offers two general equilibrium notions: Ratio–Lindahl and ratio equilibria, which respectively correspond to a mixed-ownership institutional system and a state-ownership institutional system for any number of private and public goods. These equilibrium notions include Kaneko's (1977) ratio equilibrium for economies with only one private good and the conventional Lindahl equilibrium as special cases. The approach we use to prove the existence allows not only for production technologies of state-owned firms to display some kinds of increasing returns to scale, but also provides a simple way to find the ratio–Lindahl equilibrium. *Journal of Economic Literature* Classification Numbers: 022, 025, 026.


## 1. INTRODUCTION

For the benefit approach to the allocation of public goods, a commonly used general equilibrium notion is the Lindahl equilibrium, which is a concept for private-ownership institutions. Under this equilibrium solution, production must take place at a price-taking, profit-maximizing point. This precludes the existence of an equilibrium if increasing returns to

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scale are present. Also, if profits are positive, they must be distributed in accordance with some exogenously given profit distributions. Further, when some firms in a society are owned by the government and the technologies of these firms do not display constant returns to scale (CRS), the conventional Lindahl mechanism is problematic since it is not clear how the profits should be distributed in the case of decreasing returns to scale (DRS), and the Lindahl equilibrium does not exist in the case of increasing returns to scale (IRS). This is true if the system is a state-ownership or a mixed-ownership system.[1]

Kaneko (1977) gave an alternative equilibrium notion, the ratio equilibrium, which is a modification of the Lindahl equilibrium in which consumers share the costs of producing public goods and thus it avoids the profits distribution problem. Ito and Kaneko (1981) further showed that the ratio equilibrium has the invariance property under transformation of units of public goods. They argue that this is a desirable property since public goods can be measured in several ways and do not have any commonly accepted physical units, because public goods are not actually traded in markets. Mas-Colell and Silvestre (1989, 1991) extended the notion of the ratio equilibrium by introducing another alternative equilibrium concept, the cost share equilibrium, which can be thought of as an endogenous profit distribution theory. Weber and Wiesmeth (1991) studied the equivalence of core and cost share equilibria. Tian and Li (1994a) reinterpreted the ratio equilibrium as a desirable state-ownership social choice rule. They considered incentive and information aspects of this kind of state-ownership system by using the theory of mechanism design originated by Hurwicz (1960, 1972a, 1972b, 1979a) and giving a feasible and continuous mechanism[2] which implements the ratio allocations and allows for general returns to scale. Thus, these results give a somewhat positive answer to the "socialist controversy" provoked by von Mises' skepticism as to even a theoretical feasibility of rational allocations under socialism—a debate between Misses and von Hayek, and Lange and Lerner (cf. von Hayek (1935, 1945), Lange (1936), and Lerner (1944)).

Besides the advantages of the ratio equilibrium mentioned above, it has some additional desirable properties which the Lindahl solution does not share. Since neither the ratio equilibrium nor the Lindahl equilibrium are equilibrium notions on positive grounds, any interest must be because of their normative properties. For the general returns to scale case, we see

[1] Here, state-ownership means the government owns the firms and individuals do not share profits (losses) but may need to share the cost of production. Mixed ownership implies the coexistence of privately owned firms and state-owned firms in the system.

[2] That is, outcome functions of the mechanism are continuous and result in feasible allocations not only for equilibrium messages but also for nonequilibrium messages.

at least three drawbacks in the Lindahl solution. (1) The Lindahl equilibria are, in general, not autarkically individually rational in the sense that some consumer can be better off if he/she consumes a commodity bundle that can be achieved soley by his/her endowment and technology without using other consumers' endowments.[3] (2) As Tian (1994) proved, the ratio allocation process is at least as informationally efficient as the Lindahl allocation process and further, in the presence of byproducts, it is informationally more efficient than the Lindahl allocation process. (3) The Lindahl correspondence is generally not in the core.

However, the two alternative equilibrium notions mentioned above are proposed only for the one private good case. Thus they are defined only in the context of partial equilibrium analysis, but the Lindahl equilibrium is defined in the context of general equilibrium analysis. While it is relatively easy to generalize the balanced linear cost share equilibrium to the multiple private goods case, it remains an open question proposed by Kaneko (1977, p. 135) and Ito and Kaneko (1981, p. 246) as to how one can generalize the ratio equilibrium concept to the multiple private goods case.

In this paper we offer two general equilibrium notions, ratio and ratio–Lindahl equilibria, which respectively correspond to a state-ownership institutional system and a mixed-ownership institutional system. Both yield Pareto-efficient allocations for economies with any number of private and public goods. We do so by extending Kaneko's (1977) ratio equilibrium for one-private-good economies to economies with any number of private goods, and by combining the notion of a ratio equilibrium with the notion of a Lindahl equilibrium. Therefore the ratio–Lindahl equilibrium integrates the ratio equilibrium as the subequilibrium concept corresponding to a state-ownership system and the Lindahl equilibrium as another subequilibrium concept corresponding to the private-ownership system. Thus, it allows for the coexistence of privately owned firms and state-owned firms in which the technologies of state-owned firms can have varying returns to scale at different scales of production (including IRS). Even though this generalization may be relatively straightforward, the ratio–Lindahl equilibrium concept provides a plausible alternative in the analysis of the decision-making process in a mixed ownership economy, compared to what is practiced in the real world.

In our model, only private goods (including those used as inputs) and those public goods produced by privately owned firms are traded in markets. The public goods produced by state-owned firms are not traded in markets even though the inputs used to produce these public goods must be purchased from markets. The government gives each state-owned firm

[3] For the formal definition of autarkic individual rationality, see Saijo (1991).

an exclusive franchise to produce some public good. This is a reasonable assumption because in certain situations competition may be impractical. Governments, realizing this, may allow for firms to take advantage of large economies of scale. Consumers may be required to share the costs of producing these public goods if they want to consume them. The other public goods are produced by privately owned firms according to the profit maximizing rule and sold in the competitive markets. The production possibility sets for privately owned firms are assumed to be convex, while the production possibility sets for the state-owned firms are not necessarily restricted to be convex so that state-owned firms can take advantage of large economices of scale. Consumers are self-interest and maximize their utilities under their budget constraint.

Thus, our equilibrium framework takes the Lindahl equilibrium (with private ownership), and the ratio equilibrium (with state ownership) as special cases of the ratio–Lindahl equilibrium. We prove the existence of ratio–Lindahl equilibria by showing that the existence of a ratio–Lindahl equilbrium for a mixed-ownership economy is equivalent to the existence of a Lindahl equilibrium for a transformed private-ownership economy. The main advantages of this approach, compared to the approaches used by Kaneko (1977) and Mas-Colell and Silvestre (1989), are that (1) it makes the proof of existence simple, (2) it gives a simple way to find the ratio–Lindahl equilibrium, and (3) it allows for production technologies of state-owned firms to display some kinds of increasing returns to scale. For instance, the equilibrium exists as long as utility functions of consumers and production functions of state-owned firms have Cobb–Douglas-type functions in conjunction with some regularity conditions.

The paper is organized as follows. Section 2 presents a basic mixed-ownership institutional model and defines two general equilibrium notions, the ratio–Lindahl equilibrium and the ratio equilibrium for public goods economies with any number of goods. In fact, the state-ownership system is a special case of the mixed ownership system and the ratio equilibrium is a special case of the ratio–Lindahl equilibrium. We then show that ratio–Lindahl allocations (and thus ratio allocations) are Pareto-efficient. Section 3 considers the existence of ratio–Lindahl equilibria by giving some sufficient conditions. Section 4 gives some examples which show how to use the results obtained in Section 3 to prove the existence and to find the equilibrium. Finally, Section 5 gives concluding remarks.

## 2. Model and Ratio-Lindahl Equilibria

### 2.1. *Economic Environments and a Mixed-Ownership System*

We will study a mixed-ownership institutional system with $N$ consumers, $L$ private goods, and $K$ public goods, indices $x$ denoting private and $y$ denoting public. Throughout this paper the subscript is used to index consumers or firms, and the superscript is used to index goods unless otherwise stated. We assume that there are no initial endowments of public goods, but that the public goods can be produced from private goods by $J$ privately owned firms and $T$ state-owned firms. For simplicity, we assume that the privately owned firms do not produce private goods as outputs.[4] The government (the rule of the system) gives each state-owned firm $t$ ($t = 1, \ldots, T$) an exclusive franchise to produce one public good[5] that other firms cannot produce, denoted by $y_{st} \in R_+$. The rest of the public goods, $H \equiv K - T$, are produced by $J$ privately owned firms and thus $T + H = K$. The vector of public goods produced by the $j$th privately owned firm is denoted by $y_{pj} \in R^H_+$ for $j = 1, \ldots, J$. Here we have used the subscripts $s$ and $p$ to denote state-owned firms and privately owned firms, respectively. We assume that production technologies of the $t$th state-owned firm and the $j$th privately owned firm are represented by production functions $f_{st}(v_{st})$ and $f_{pj}(v_{pj})$, where $v_{st}, v_{pj} \in \mathbb{R}^L_+$ are inputs used by the $t$th state-owned firm and the $j$th privately owned firm, respectively. We assume throughout that, for each state-owned firm $t$, $f_{st}(0) = 0$ and $f_{st}$ is nondecreasing and continuous for all $t = 1, \ldots, T$; for each privately owned firm $j$, $f_{pj}(0) = 0$ and $f_{pj}$ is nondecreasing and continuous, and furthermore displays nonincreasing returns to scale (i.e., the production set specified by $f_{pj}$ is convex).[6] Thus, while the production possibility sets of privately owned firms are assumed to be convex, the production possibility sets of state-owned firms are not necessarily restricted to be convex. So we allow for the technologies of state-owned firms but not privately owned firms to display IRS. Let $f_s(v_s) = (f_{s1}(v_{s1}), \ldots, f_{sT}(v_{sT}))$, $f_p(v_p) = (f_{p1}(v_{p1}), \ldots, f_{pJ}(v_{pJ}))$, and $f(v) = (f_s(v_s), f_p(v_p))$, and $v_s = (v_{s1}, \ldots, v_{sT})$, $v_p = (v_{p1}, \ldots, v_{pJ})$, and $v = (v_s, v_p)$. Denote by $\bar{y}_p = \sum_{j=1}^{J} y_{pj}$ aggregate public goods produced by privately owned firms.

Each consumer $i$'s characteristic is denoted by $e_i = (w_i, u_i)$, where $w_i \geq 0$ [7] is the initial endowment of private goods and $u_i(x_i, y_s, \bar{y}_p)$ is the utility function for amounts $x_i \in \mathbb{R}^L_+$ of private goods and $(y_s, \bar{y}_p) \in \mathbb{R}^K_+$ of public goods. Here $u_i(\cdot)$ is assumed to be continuous, strictly increasing in, say, the first component of private goods (which can be regarded as a numéraire), and nondecreasing in other components (i.e., no private or

[4] Extension to the case of producing private goods is straightforward.

[5] The results of this paper still hold even if there are byproducts in production of the state-owned firm (cf. Tian (1994)). This case is more general and realistic than the case of each firm producing only one public good.

[6] A sufficient condition for $f_{pj}$ to display nonincreasing returns to scale is that $f_{pj}$ is concave. Note that the quasi-concavity of $f_{pj}$ does not guarantee that $f_{pj}$ displays nonincreasing returns to scale.

[7] As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $l = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_l > b_l$ for all $l = 1, \ldots, m$.

public bads). Each consumer $i$ has $J$ nonnegative profit shares $\theta_{ij} \in [0, 1]$ from $J$ privately owned firms, satisfying $\sum_{i=1}^{N} \theta_{ij} = 1, j = 1, \ldots, J$. A mixed-ownership economy is the full vector $e = (e_1, \ldots, e_N, f(\cdot), \{\theta_{ij}\})$, and the set of all such economies is denoted by $E$.

We further assume that only private goods (including those used as inputs) and those public goods $y_p$ produced by privately owned firms are traded in markets. Their market prices are denoted by $p \in \mathbb{R}^L_+$ and $q \in \mathbb{R}^H_+$, respectively. The public goods produced by state-owned firms are not traded in markets even though the inputs used to produce these public goods must be purchased from markets. Consumers must share the costs of product to consume these public goods. The cost functions of the $t$th state-owned firm and the $j$th privately owned firm which are dual to production functions $f_{st}$ and $f_{pj}$ are denoted by $C_{st}$ and $C_{pj}$, respectively. Let $C_s(p, y_s) = (C_{s1}(p, y_{s1}), \ldots, C_{sT}(p, y_{sT}))$ and $C_p(p, y_p) = (C_{p1}(p, y_{p1}), \ldots, C_{pJ}(p, y_{pJ}))$, where $y_s = (y_{s1}, \ldots, y_{sT})$ and $y_p = (y_{p1}, \ldots, y_{pJ})$. Denote by $r_i \in \mathbb{R}^T_+$ and $q_i \in \mathbb{R}^H_+$ $(i = 1, \ldots, N)$ the ratio vectors for the costs of state-owned firms and the personalized price vectors for public goods produced by privately owned firms, respectively.

An allocation of the economy $e$ is a vector $(x, y) \equiv (x, y_s, y_p) \in R^{NL+T+JH}_+$. An allocation $(x, y)$ is *feasible* if there is $v = (v_s, v_p) \in R^{(T+J)L}_+$ such that $y \leqq f(v)$ and the resource constraints

$$\sum_{i=1}^{N} x_i + \sum_{t=1}^{T} v_{st} + \sum_{j=1}^{J} v_{pj} \leqq \sum_{i=1}^{N} w_i \tag{1}$$

are satisfied. An allocation $(x, y)$ is *Pareto optimal* if it is feasible and there is no other feasible allocation $(x', y')$ such that $u_i(x_i', y_s', \tilde{y}_p') \geqq u_i(x_i, y_s, \tilde{y}_p)$ for all $i = 1, \ldots, N$, with strict inequality for at least one $i$.

*Remark* 1. For any feasible allocation $(x, y)$ and price vector $p \in R^L_+$, by the facts that $p \cdot v_{st} \geqq C_{st}(p, y_{st})$ and $p \cdot v_{pj} \geqq C_{pj}(p, y_{pj})$, we have from (1) that

$$\sum_{i=1}^{N} p \cdot x_i + \iota_T \cdot C_s(p, y_s) + \iota_J \cdot C_p(p, y_p) \leqq \sum_{i=1}^{N} p \cdot w_i, \tag{2}$$

where $\iota_m$ is a vector of ones of dimension $m$.

### 2.2. *Ratio-Lindahl and Ratio Equilibria*

DEFINITION 1. A *ratio–Lindahl equilibrium* corresponding to a mixed-ownership economy $e$ is a price–ratio system $(p^*, r_1^*, \ldots, r_N^*, q_1^*, \ldots, q_N^*) \in R^{L+NK}_+$ and a feasible allocation $(x^*, y^*)$ such that

(1) $p^* \cdot x_i^* + r_i^* \cdot C_s(p^*, y_s^*) + q_i^* \cdot \bar{y}_p^* \leqq p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)]\ \forall i = 1, \ldots, N$;

(2) $u_i(x_i, y_s, \bar{y}_p) > u_i(x_i^*, y_s^*, \bar{y}_p^*)$ implies $p^* \cdot x_i + r_i^* \cdot C_s(p^*, y_s) + q_i^* \cdot \bar{y}_p > p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)]\ \forall i = 1, \ldots, N$;

(3) $q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*) = \max_{y_{pj} \in R_+^H}(q^* \cdot y_{pj} - C_{pj}(p^*, y_{pj}))\ \forall j \in \{1, \ldots, J\}$,

where $\sum_{i=1}^{N} r_i^* = \iota_T$ and $q^* = \sum_{i=1}^{N} q_i^*$. We will call $(x^*, y^*)$ a *ratio–Lindahl allocation*. Denote by RL($e$) the set of all such allocations.

Let $\nu(p, y) = (\nu_s(p, y_s), \nu_p(p, y_p))$ be the conditional input demand correspondence for producing $y$ units of public goods at price vector $p$.[8] The following fact shows that $v^* \in \nu(p^*, y^*)$ if $v^*$ is an input vector associated with a ratio–Lindahl allocation $(x^*, y^*)$ with a price-ratio system$(p^*, r_1^*, \ldots, r_N^*, q^*, q_1^*, \ldots, q_N^*)$.

*Fact* 1. If an allocation $(x^*, y^*)$ is a ratio–Lindahl allocation with a price-ratio system $(p^*, r_1^*, \ldots, r_N^*, q_1^*, \ldots, q_N^*)$ and if $v^*$ is an input vector such that $(x^*, y^*)$ is feasible, then $v^* \in \nu(p^*, y^*)$.

*Proof.* Since $(x^*, y^*)$ is a ratio–Lindahl allocation, by the budget constraints of consumers and the monotonicity of preferences, we have

$$\sum_{i=1}^{N} p^* \cdot x_i^* + \iota_T \cdot C_s(p^*, y_s^*) + \iota_J \cdot C_p(p^*, y_p^*) = \sum_{i=1}^{N} p^* \cdot w_i. \qquad (3)$$

Since $v^*$ is an input vector such that $(x^*, y^*)$ is feasible, we then have

$$\sum_{i=1}^{N} p^* \cdot x_i^* + \sum_{t=1}^{T} p^* \cdot v_{st}^* + \sum_{j=1}^{J} p^* \cdot v_{pj}^* \leqq \sum_{i=1}^{N} p^* \cdot w_i. \qquad (4)$$

We show that $v^* \in \nu(p^*, y^*)$. Suppose not. Then

$$\iota_T \cdot C_s(p^*, y_s^*) + \iota_J \cdot C_p(p^*, y_p^*) < \sum_{t=1}^{T} p^* \cdot v_{st}^* + \sum_{j=1}^{J} p^* \cdot v_{pj}^*$$

and thus, by (3), we have

[8] Note that, in general, $C$ is not differentiable and $\nu$ is not single-valued and/or continuous. This is particularly true for cost functions of state-owned firms which have IRS technologies. However, in the case where $C(p, y)$ is differentiable with respect to $p \in \mathbb{R}_{++}^L$, $\nu(p, y)$ becomes single-valued and is given by $\nu(p, y) = \partial C(p, y)/\partial p$ by Shephard's lemma.

$$\sum_{i=1}^{N} p^* \cdot w_i = \sum_{i=1}^{N} p^* \cdot x_i^* + \iota_T \cdot C_s(p^*, y_s^*) + \iota_J \cdot C_p(p^*, y_p^*)$$
$$< \sum_{i=1}^{N} p^* \cdot x_i^* + \sum_{t=1}^{T} p^* \cdot v_{st}^* + \sum_{j=1}^{J} p^* \cdot v_{pj}^*, \quad (5)$$

which contradicts (4). Q.E.D.

If there are no privately owned firms, we have the following equilibrium notion, which is a special case of the ratio–Lindahl equilibrium and generalizes the ratio equilibrium of Kaneko (1977) to the case of multiple private goods.

DEFINITION 2. A *ratio equilibrium* corresponding to a state-ownership economy $(e_1, \ldots, e_N, f_s)$ is a price-ratio system $(p^*, r_1^*, \ldots, r_N^*) \in R_+^{L+NT}$ and a feasible allocation $(x^*, y_s^*)$ such that

(1) $p^* \cdot x_i^* + r_i^* \cdot C_s(p^*, y_s^*) \leqq p^* \cdot w_i, \forall i = 1, \ldots, N$;

(2) $u_i(x_i, y_s) > u_i(x_i^*, y_s^*)$ implies $p^* \cdot x_i + r_i^* \cdot C_s(p^*, y_s) > p^* \cdot w_i$, $\forall i = 1, \ldots, N$,

where $\sum_{i=1}^{N} r_i^* = \iota_T$. We will call $(x^*, y_s^*)$ a *ratio allocation*. Denote by $R(e)$ the set of all such allocations.

If there are no state-owned firms, we have the conventional Lindahl equilibrium, which is a special case of the ratio–Lindahl equilibrium.

DEFINITION 3. A *Lindahl equilibrium* corresponding to a private-ownership economy $(e_1, \ldots, e_N, f_p, \{\theta_{ij}\})$ is a price system $(p^*, q_1^*, \ldots, q_N^*) \in R_+^{L+NK}$ and a feasible allocation $(x^*, y_p^*)$ such that

(1) $p^* \cdot x_i^* + q_i^* \cdot \bar{y}_p^* \leqq p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)], \forall i = 1, \ldots, N$;

(2) $u_i(x_i, \bar{y}_p) > u_i(x_i^*, \bar{y}_p^*)$ implies $p^* \cdot x_i + q_i^* \cdot \bar{y}_p > p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)], \forall i = 1, \ldots, N$;

(3) $q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*) = \max_{y_{pj} \in R_+^H}(q^* \cdot y_{pj} - C_{pj}(p^*, y_{pj})), \forall j \in \{1, \ldots, J\}$,

where $q^* = \sum_{i=1}^{N} q_i^*$. We will call $(x^*, y_p^*)$ a *Lindahl allocation*. Denote by $L(e)$ the set of all such allocations.

*Remark* 2. Note that when production functions of all state-owned firms display CRS, $C_s(p, y_s)$ becomes linear in $y_s$. As a consequence, $(x^*, y^*)$ is a ratio–Lindahl allocation with a price-ratio system $(p^*,$

$r_1^*, \ldots, r_N^*, q_1^*, \ldots, q_N^*) \in R_+^{L+NK}$ if and only if it is a Lindahl allocation with a price system $(p^*, r_1^* C_s(p^*, \iota_T), \ldots, r_N^* C_s(p^*, \iota_T), q_1^*, \ldots, q_N^*) \in R_+^{L+NK}$. Thus, for the case where technologies of all state-owned firms display CRS, the existence of the ratio–Lindahl equilibrium is guaranteed by the existence of the Lindahl equilibrium.

Thus, the ratio–Lindahl equilibrium which corresponds to a mixed-ownership economy is the combination of the ratio equilibrium which corresponds to a state-ownership system and the Lindahl equilibrium which corresponds to a private-ownership system. Hence the system reduces to state ownership when $J = 0$ and to private ownership when $T = 0$ or $C_s$ is linear in $y_s$. Also, the ratio–Lindahl equilibrium reduces to the ratio equilibrium of Kaneko (1977) when $J = 0$ and $L = 1$. So the ratio–Lindahl equilibrium unifies these equilibrium concepts. Further, the ratio–Lindahl equilibrium coincides with the balanced linear cost share equilibrium of Mas-Colell and Silvestre (1989) when $J = 0$ and $K = 1$.

Since the ratio equilibrium is a special case of the ratio–Lindahl equilibrium, in the following we only consider the properties and existence of the ratio–Lindahl equilibrium. The results for the ratio equilibrium can be obtained as corollaries.

PROPOSITION 1. *If $(x, y)$ is a ratio–Lindahl allocation, then it is Pareto optimal.*

*Proof.* Suppose not; then there exists a feasible allocation $(x', y')$ such that there is at least one consumer (say, consumer $i$) that is better off and others are not worse off under $(x', y')$. Then we must have

$$p^* \cdot x_i^* + r_i^* \cdot C_s(p^*, y_s^*) + q_i^* \cdot \bar{y}_p' > p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)]$$

and

$$p^* \cdot x_k' + r_k^* \cdot C_s(p^*, y_s') + q_k^* \cdot \bar{y}_p' \geqq p^* \cdot w_k + \sum_{j=1}^{J} \theta_{kj}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)]$$

for all $k$ by monotonicity of preferences. Therefore, adding up these inequalities, we have

$$\sum_{k=1}^{N} p^* \cdot w_k + q^* \cdot \bar{y}_p^* - \iota_J \cdot C_p(p^*, y_p^*)$$
$$< \sum_{k=1}^{N} p^* \cdot x_k' + \iota_T \cdot C_s(p^*, y_s') + q^* \cdot \bar{y}_p' \tag{6}$$
$$\leqq \sum_{k=1}^{N} p^* \cdot w_k + q^* \cdot \bar{y}_p' - \iota_J \cdot C_p(p^*, y_p'),$$

where the last inequality comes from Eq. (2). Thus

$$q^* \cdot \bar{y}_p^* - \iota_J \cdot C_p(p^*, y_p^*) < q^* \cdot \bar{y}_p' - \iota_J \cdot C_p(p^*, y_p'), \tag{7}$$

which contradicts the fact that $y_p^*$ is the profit maximizing level of outputs. Q.E.D.

## 3. The Existence of Ratio–Lindahl Equilibria

This section considers the existence of ratio–Lindahl equilibria. Note that we cannot follow the approaches used by Kaneko (1977) or Mas-Colell and Silvestre (1989) to prove the existence of the ratio–Lindahl equilibrium since these approaches require that utility functions be quasi-concave and production technologies of state-owned firms display no IRS. Here we use another approach which shows that proving the existence of a ratio–Lindahl equilibrium for the original mixed-ownership economy is equivalent to proving the existence of a Lindahl equilibrium for the transformed private-ownership economy. Thus the existence of a ratio–Lindahl equilibrium is a consequence of a Lindahl equilibrium for the transformed economy.[9] The main advantage of this approach is that we can allow the production technologies of state-owned firms to display some kinds of IRS and the ratio–Lindahl equilibrium to still exist. By a transformation, we linearize the cost functions of state-owned firms and thus reduce the mixed-ownership economy to a private-ownership economy. We can thus easily prove the existence of ratio–Lindahl equilibria by using the results on Lindahl equilibria in the existing literature, such as those in Foley (1970), Milleron (1972), and Roberts (1974).

For a mixed-ownership economy $e$ with $e_i = (w_i, u_i)$ and production functions of state-owned firms $y_s = f_s(v_s) = (f_{s1}(v_{s1}), \ldots, f_{sT}(v_{sT}))$ which may display CRS, DRS, or IRS, we can define a transformed private-ownership economy $e'$ with $e_i' = (w_i, u_i')$, where $u_i'(x_i, v_s, \bar{y}_p) \equiv u_i(x, f_s(v_s),$

[9] This approach is somewhat similar to the one used by Foley (1970) and Milleron (1972) to prove the existence of the Lindahl equlibrium by transforming a public goods economy to a private goods economy and then proving the existence of a Walrasian equilibrium.

$\bar{y}_p$) with $x$ being private goods and $(v_s, \bar{y}_p) \in \mathbb{R}_+^{TL+H}$ being public goods. Note that in this transformed economy we treat the inputs used in the production of public goods by the state-owned firms as public "consumption" goods. Thus, the market price vector of public "consumption" goods $v_s$ is the same as the prices of private goods since they are the "same" goods. Also, the budget constraint becomes $p \cdot x_i + \sum_{t=1}^{T} \gamma_{it} \cdot v_{st} + q_i \cdot \bar{y}_p \leqq p \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q \cdot y_{pj}^* - C_{pj}(p, y_{pj}^*)]$, where $\gamma_{it}$ is the personalized price vector of $v_{st}$. The transformed budget constraint is linear in all variables, including $v_s$. Thus, we can think of $v_s$ as public goods with CRS production technologies for the transformed economy. Propositions 2 and 3 below show that, under some standard conditions, finding a ratio–Lindahl equilibrium is equivalent to finding a Lindahl equilibrium for the transformed economy.

PROPOSITION 2. *For any mixed-ownership economy e, if* $(x^*, y^*)$ *is a ratio–Lindahl allocation with a price-ratio system* $(p^*, r_1^*, \ldots, r_N^*, q_1^*, \ldots, q_N^*) \in R_+^{L+NK}$ *and with the input vector* $(v_s^*, v_p^*)$, *then* $(x^*, z^*) \equiv (x^*, v_s^*, y_p^*)$ *is a Lindahl allocation with a price system* $(p^*, \gamma_1^*, \ldots, \gamma_N^*, q_1^*, \ldots, q_N^*)$ *for the transformed private-ownership economy e′, where* $\gamma_{it}^* = r_{it}^* p^*$ *for all* $t = 1, \ldots, T$.

*Proof.* We need to show that if $(x^*, y^*)$ is a ratio–Lindahl allocation for $e$, then $(x^*, z^*) = (x^*, v_s^*, y_p^*)$ is a Lindahl allocation for $e'$. Since $(x^*, z^*)$ is feasible by Fact 1 and $y_p^*$ is the profit maximizing level of outputs, we only need to show that all consumers are maximizing their utilities. Suppose, by way of contradiction, that there is some $i$ and $(x_i, v_s, \bar{y}_p)$ such that $p^* \cdot x_i + \sum_{t=1}^{T} \gamma_{it}^* \cdot v_{st} + q_i^* \cdot \bar{y}_p \leqq p^* \cdot w_i + \sum_{j=1}^{J} \theta_{pj}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)]$ and $u_i(x_i, f_s(v_s), \bar{y}_p) > u_i(x_i^*, f_s(v_s^*), \bar{y}_p^*)$. Let $y_s = f_s(v_s)$. Then $u_i(x_i, y_s, \bar{y}_p) > u_i(x_1^*, y_s^*, \bar{y}_p^*)$ and yet $p^* \cdot x_i + r_i^* \cdot C_s(p^*, y_s) + q_i^* \cdot \bar{y}_p \leqq p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)]$ since $p^* \cdot v_{st} \geqq C_{st}(p^*, y_{st})$.[10] But this contradicts the hypothesis that $(x^*, y^*)$ is a ratio–Lindahl allocation. Q.E.D.

Before proving the converse of Proposition 2, we first prove a lemma. Although the lemma requires the assumptions that utility functions and production functions of state-owned firms are differentiable, the differentiability assumption will be removed from Theorem 1 below. Theorem 1 gives the existence of the ratio–Lindahl equilibrium by using the approximation theory of Mas-Colell (1974, 1985). The approximation approach is now standard and the shortest way to prove a theorem in a continuity framework via smoothing and differentiability methods.[11]

[10] Note that $r_i^*$, by the definition, satisfies $\gamma_{it}^* = r_{it}^* p^*$ for all $t = 1, \ldots, T$.

[11] For instance, Mas-Colell and Silvestre (1989, Proposition 7) used this approach to prove their equilibrium existence theorem.

LEMMA 1. *For any mixed-ownership economy e, suppose that $u_i$ and $f_s$ are differentiable and the composite functions $u_i'(x_i, v_s, \tilde{y}_p) \equiv u_i(x, f_s(v_s), \tilde{y}_p)$ are quasiconcave. If $(p^*, \gamma_1^*, \ldots, \gamma_N^*, q_1^*, \ldots, q_N^*)$ is a price system corresponding to the Lindahl allocation $(x^*, v_s^*, y_p^*)$ for the transformed private-ownership economy e', then $(p^*, \tilde{\gamma}_1^*, \ldots, \tilde{\gamma}_N^*, q_1^*, \ldots, q_N^*)$ is also a price system corresponding to the same Landahl allocation $(x^*, v_s^*, y_p^*)$ for the economy e', where $\tilde{\gamma}_{it}^* = r_{it}^* p^*$ with*

$$r_{it}^* = \begin{cases} \dfrac{\mu_{it}^*}{\sum_{k=1}^N \mu_{kt}^*} & \text{if } \sum_{k=1}^N \mu_{kt}^* > 0 \\ \dfrac{1}{N} & \text{otherwise} \end{cases} \tag{8}$$

*for all $t = 1, \ldots, T$. Here*

$$\mu_{it}^* = \begin{cases} \dfrac{\partial u_i(x^*, y_s^*, \tilde{y}_p^*)/\partial y_{st}}{\lambda_i^*} & \text{if } \lambda_i^* > 0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

*for all $t = 1, \ldots, T$, where $\lambda_i^*$ is the optimal Lagrange multiplier obtained by maximizing consumer i's Lagrange function:*

$$L_i(x_i, v_s, \tilde{y}_p, \lambda_i) \equiv u_i(x_i, f_s(v_s), \tilde{y}_p) + \lambda_i[p^* \cdot w_i + \sum_{j=1}^J \theta_{ij}(q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*))$$
$$-p^* \cdot x_i - \sum_{t=1}^T \gamma_{it}^* \cdot v_{st} - q_i^* \cdot \tilde{y}_p]. \tag{10}$$

*Proof.* Since $(x^*, v_s^*, y_p^*)$ is a Lindahl allocation for the transformed economy $e'$, each consumer is maximizing his/her utility. Then, differentiating (10) with respect to $v_{st}^l$ and using the Kuhn–Tucker first-order condition, we have

$$\frac{\partial u_i(x^*, y_s^*, \tilde{y}_p^*)}{\partial y_{st}} \frac{\partial f_{st}(v_{st}^*)}{\partial v_{st}^l} \leqq \lambda_i^* \gamma_{it}^{*l} \tag{11}$$

with equality if $v_{st}^{*l} > 0$ for all $t = 1, \ldots, T$ and $l = 1, \ldots, L$. Note that when $\lambda_i^* = 0$, $\partial u_i(x^*, y_s^*, \tilde{y}_p^*)/\partial y_{st} = 0$ and thus Eq. (11) reduces to $0 \leqq \gamma_{it}^{*l}$ since $u_i$ is nondecreasing.

We first consider the case of $\sum_{k=1}^{N} \mu_{kt}^* > 0$. Dividing both sides of Eq. (11) by $\lambda_i^*$ for those $\lambda_i^* > 0$ and then summing over consumers $i$, we have

$$\frac{\partial f_{st}(v_{st}^*)}{\partial v_{st}^l} \sum_{k=1}^{N} \mu_{kt}^* \leqq p^{*l}. \tag{12}$$

Here we have used the facts that $\sum_{i=1}^{N} \gamma_{it}^* = p^*$ (the sum of personalized prices equals the market prices for public "consumption" goods $v_s$) and $0 \leqq \gamma_{it}^{*l}$ for those $i$ with $\lambda_i^* = 0$. Dividing both sides of (12) by $\sum_{k=1}^{N} \mu_{kt}^*$ and multiplying by $\partial u_i(x^*, y_s^*, \bar{y}_p^*)/\partial y_{st}$, we obtain

$$\begin{aligned} \frac{\partial u_i(x^*, y_s^*, \bar{y}_p^*)}{\partial y_{st}} \frac{\partial f_{st}(v_{st}^*)}{\partial v_{st}^l} &\leqq \left[\frac{\partial u_i(x^*, y_s^*, \bar{y}_p^*)/\partial y_{st}}{\sum_{k=1}^{N} \mu_{kt}^*}\right] p^{*l} \\ &= \lambda_i^* r_{it}^* p^{*l} = \lambda_i^* \bar{\gamma}_{it}^* \end{aligned} \tag{13}$$

for all $l = 1, \ldots, L$. Thus $v_s^*$ satisfies the Kuhn–Tucker first-order conditions for the vector of personalized prices $\bar{\gamma}_i^*$.

Now consider the case of $\sum_{k=1}^{N} \mu_{kt}^* = 0$. Then either $\lambda_i^* = 0$ or $\partial u_i(x^*, y_s^*, \bar{y}_p^*)/\partial y_{st} = 0$ for all $i = 1, \ldots, N$. In both cases $v_s^*$ clearly satisfies the first-order equation (11) for the personalized prices $\bar{\gamma}_{it}^* = 1/N p^*$.

Finally, since the $u_i'$ are quasiconcave and the Kuhn–Tucker first-order conditions are also satisfied for $(x_i^*, \bar{y}_p^*)$ under prices $(p^*, q_i^*)$, all consumers are maximizing their utilities at $(x^*, v_s^*, \bar{y}_p^*)$ under the price system $(p^*, \bar{\gamma}_1^*, \ldots, \bar{\gamma}_N^*, q_1^*, \ldots, q_N^*)$. Also, allocation $(x^*, v_s^*, y_p^*)$ is feasible and $y_p^*$ is the profit maximizing level of outputs, therefore $(p^*, \bar{\gamma}_1^*, \ldots, \bar{\gamma}_N^*, q_1^*, \ldots, q_N^*)$ is a price system corresponding to the Lindahl allocation $(x^*, v_s^*, y_p^*)$ for the transformed economy $e'$. Q.E.D.

*Remark* 3. It can be easily shown that if $v_{st}^{*l} \in \mathbb{R}_{++}$ and $\lambda_i^* > 0$ for all $i$, the personalized price $\gamma_{it}^{*l}$ is uniquely determined and given by $\gamma_{it}^{*l} = r_{it}^* p^{*l}$ with

$$r_{it}^* = \frac{[\partial u_i(x^*, y_s^*, \bar{y}_p^*)/\partial y_{st}]/\lambda_i^*}{\sum_{k=1}^{N} [\partial u_k(x^*, y_s^*, \bar{y}_p^*)/\partial y_s^t]/\lambda_k^*},$$

provided $\sum_{k=1}^{N} (\partial u_k(x^*, y_s^*, \bar{y}_p^*)/\partial y_{st}) > 0$. Thus, $r_{it}^* = \gamma_{it}^{*l}/p^{*l}$ if $p^{*l} > 0$. In particular, $r_{it}^*$ is given by $r_{it}^* = \gamma_{it}^{*l}$ when the price of the first component of private goods is normalized to be one. This gives a simple way to find the ratios of the ratio–Lindahl equilibrium from the Lindahl prices. (See Example 1 below.)

**Proposition** 3. *For any mixed-ownership economy e, suppose that* $u_i$ *and* $f_s$ *are differentiable and the composite functions* $u_i'(x_i, v_s, \tilde{y}_p) \equiv u_i(x, f_s(v_s), \tilde{y}_p)$ *are quasiconcave. If* $(x^*, v_s^*, y_p^*)$ *is a Lindahl allocation with a price system* $(p^*, \gamma_1^*, \ldots, \gamma_N^*, q_1^*, \ldots, q_N^*)$ *for the transformed private-ownership economy* $e'$, *then* $(x^*, y^*) \equiv (x^*, f_s(v_s^*), y_p^*)$ *is a ratio–Lindahl allocation with the price-ratio system* $(p^*, r_1^*, \ldots, r_N^*, q_1^*, \ldots, q_N^*)$ *for the economy e, where* $r_{it}^*$ *is given by (8).*

*Proof.* First note that by Lemma 1, if the allocation $(x^*, v_s^*, y_p^*)$ with the price system $(p^*, \gamma_1^*, \ldots, \gamma_N^*, q_1^*, \ldots, q_N^*)$ is a Lindahl equilibrium for the transformed economy $e'$, $(x^*, v_s^*, y_p^*)$ with the price system $(p^*, \bar{\gamma}_1^*, \ldots, \bar{\gamma}_N^*, q_1^*, \ldots, q_N^*)$ is also a Lindahl equilibrium for the transformed economy $e'$, where $\bar{\gamma}_{it}^* = r_{it}^* p^*$ for all $t = 1, \ldots, T$.

Next we show that $(x^*, y^*) = (x^*, f_s(v_s^*), y_p^*)$ is a ratio–Lindahl allocation with the price-ratio system $(p^*, r_1^*, \ldots, r_N^*, q_1^*, \ldots, q_N^*)$ for $e$. Similar to the proof of Proposition 2, we only need to show that each consumer $i$ is maximizing his/her utility. Suppose, by way of contradiction, that there is a consumer $i$ and $(x_i, y_s, \tilde{y}_p)$ such that $p^* \cdot x_i + r_i^* \cdot C_s(p^*, y_s) + q_i^* \cdot \tilde{y}_p \leqq p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(p^*, y_{pj}^*)]$ and $u_i(x_i, y_s, \tilde{y}_p) > u_i(x_i^*, y_s^*, \tilde{y}_p^*)$. Let $v_s \in \nu_s(p^*, y_s)$. Then $f_s(v_s) = y_s$ and $u_i(x_i, f_s(v_s), \tilde{y}_p) > u_i(x_i^*, f_s(v_s^*), \tilde{y}_p^*)$. Hence we have $u_i'(x_i, v_s, \tilde{y}_p) > u_i'(x_i^*, v_s^*, \tilde{y}_p^*)$ and $p^* \cdot x_i + \sum_{t=1}^{T} r_{it}^*(p^* \cdot v_{st}) + q_i^* \cdot \tilde{y}_p = p^* \cdot x_i + \sum_{t=1}^{T} \bar{\gamma}_{it}^* \cdot v_{st} + q_i^* \cdot \tilde{y}_p \leqq p \cdot w_i + \sum_{j=1}^{J} \theta_{ij}[q^* \cdot y_{pj}^* - C_{pj}(y_{pj}^*)]$. But this contradicts the hypothesis that $(x^*, v_s^*, y_p^*)$ is a Lindahl allocation. Q.E.D.

*Remark* 4. Note that in the proof of Proposition 3, we need to assume that $u_i$ and $f_s$ are differentiable and $u_i'$ is quasi-concave. However, from the proof of Proposition 3, we can see that if there is only one private good or if the personalized price vector $\gamma_i$ has the form of $\gamma_{it}^* = a_{it}^* p^*$ for all $i$ with $\sum_{i=1}^{N} a_{it}^* = 1$ for the multiple private goods case, then the conclusion of Proposition 3 still holds without making these assumptions (cf. Tian and Li (1990b)).

*Remark* 5. Proposition 3, even though simple, is very useful in proving the existence of ratio–Lindahl equilibria, especially when some of the state-owned firms have IRS technologies, in which case one cannot use the standard techniques in the literature to prove the existence. It also provides a simple way to find the ratio–Lindahl equilibrium by finding the Lindahl equilibrium for a transformed economy. After obtaining the existence or solution of Lindahl equilibria for $e'$, we tranform them back to obtain ratio–Lindahl equilibria for $e$.

Now we use Proposition 3 to prove the existence of ratio–Lindahl equilibria.

THEOREM 1. *For any mixed-ownership economy e, if the composite functions* $u_i'(x_i, v_s, \bar{y}_p) \equiv u_i(x, f_s(v_s), \bar{y}_p)$ *are quasiconcave, then there exists a ratio–Lindahl equilibrium for the economy e.*

*Proof.* We first consider the case where utility functions $u_i$ and production function $f_s$ are differentiable. For the transformed economy $e'$ which corresponds to a private-ownership economy with CRS technologies for the "public goods" $v_s$, since $u_i'$ is continuous and quasiconcave, $f_p$ is continuous and displays nonincreasing returns to scale. All the conditions in Theorem 3.1 of Milleron (1972, p. 443) are satisfied and thus we know the existence of a Lindahl equilibrium. Thus by Proposition 3, there exists a ratio–Lindahl equilibrium for the economy $e$.

We now consider the case where utility and production functions are continuous but not necessarily differentiable. By the approximation theory of Mas-Colell (1974, 1985), we know that smooth economies are dense in all continuous economies and thus we can approximate non-smooth economies by smooth ones. Since ratio–Lindahl equilibria, as we have shown above, exist for the smooth economies and the equilibrium correspondence RL is closed (i.e., its graph is closed) by Berge's maximum theorem, any economy $e$ satisfying conditions imposed in the paper is a limit of a sequence of smooth economies and thus RL($e$) $\neq \emptyset$ by the closedness of the correspondence. Thus a ratio–Lindahl equilibrium exists. Q.E.D.

Since the ratio equilibrium is a special case of the ratio–Lindahl equilibrium, the above theorem actually proves the existence of ratio equilibria. We state it here as a corollary.

COROLLARY 1. *For any state-ownership economy* $(e_1, \ldots, e_N, f_s)$, *if the composite functions* $u_i'(x_i, v_s) \equiv u_i(x, f_s(v_s))$ *are quasi-concave, then there exists a ratio equilibrium for this economy.*

The proof of the existence of ratio–Lindahl equilibria in Theorem 1 only requires the quasiconcavity of the transformed utility functions $u_i'$. The following result gives a sufficient condition for the composite functions $u_i'$ to be quasi-concave.

*Fact* 2. If $u_i(x_i, y_s, \bar{y}_p)$ is quasi-concave and $y_s = f(v_s)$ is concave, then $u_i'(x_i, v_s, \bar{y}_p)$ is quasi-concave in $(x_i, v_s, y_p)$.

*Proof.* For $x_i^1, x_i^2, v_s^1, v_s^2 \in R_+^L$, $y^1, y^2 \in R_+^K$ and $y_s^k = f_s(v_s^k)$ $(k = 1, 2)$, and $\lambda \in [0, 1]$. Let $x_i^\lambda = \lambda x_i^1 + (1 - \lambda)x_i^2$ and define $y^\lambda$, $v_s^\lambda$ similarly. Define $f_s^\lambda = \lambda f_s(v_s^1) + (1 - \lambda)f(v_s^2)$. Then we have $u_i'(x_i^\lambda, v_s^\lambda, \bar{y}_p^\lambda) = u_i(x_i^\lambda, f_s(v_s^\lambda), \bar{y}_p^\lambda) \geqq u_i(x_i^\lambda, f_s^\lambda, \bar{y}_p^\lambda) \geqq \min_{\{k=1,2\}}\{u_i(x_i^k, y_s^k, \bar{y}_p^k)\} = \min_{\{k=1,2\}} u_i'(x_i^k, v_s^k, \bar{y}_p^k)$. Therefore $u_i'(\cdot)$ is quasiconcave. The first inequality comes from the facts that $f_s$ is concave and $u_i$ is nondecreasing, and the second inequality comes from the quasiconcavity of $u_i$. Q.E.D.

From Theorem 1 and Fact 2, we immediately have the following corollary which contains the result of Kaneko (1977) as a special case.

COROLLARY 2. *For an economy e, if utility functions $u_i$ are quasi-concave and production functions $f_s$ are concave, then there exists a ratio–Lindahl equilibrium for the economy e.*

The next proposition shows that even if the production functions $f_s$ displays IRS, utility functions of consumers after transformation can be quasi-concave and thus by Theorem 1 we know the existence of ratio–Lindahl equilibria.

We first need a lemma.

LEMMA 2. *If a function F is the Cobb–Douglas-type function (i.e., it is of the form $F(z_1, z_2, \ldots, z_m) = z_1^{\alpha 1} z_2^{\alpha 2} \cdots z_M^{\alpha M}$ for $z \in R_+^M$ and $(\alpha_1, \alpha_2, \ldots, \alpha_M) \in \mathbb{R}_+^M$), then $F(\cdot)$ is quasi-concave.*

*Proof.* See the proof in Berge (1963, p. 209).

PROPOSITION 4. *If both utility functions $u_i$ and production functions $f_s$ are of the Cobb–Douglas-type ($f_s$ can be any degree of IRS as long as it is a Cobb–Douglas-type function), then the composite functions $u_i'(x_i, v_s, \bar{y}_p) \equiv u_i(x, f_s(v_s), \bar{y}_p)$ are quasi-concave.*

*Proof.* Since the transformed utility functions $u_i'$ are still of the Cobb–Douglas-type, they are quasi-concave by Lemma 2.

## 4. EXAMPLES

In this section we present two examples which illustrate how to find ratio equilibria using the results obtained in Section 3. The first example uses Proposition 3 to find a ratio–Lindahl equilibrium solution from the transformed Lindahl equilibrium. Note that the production function $f_s$ in Example 1 displays IRS.

EXAMPLE 1. Consider an economy with $N$ consumers who consume two private goods $x = (x^1, x^2)$ and one public good $y_s \equiv y$. For simplicity, we assume $J = 0$, i.e., there is no privately owned firm. So we will drop the subscript $s$ below. The production function is given by $y = v^1 v^2$ which displays IRS. Suppose that utility functions are Cobb–Douglas functions:

$$u_i(x_i^1, x_i^2, y) = (x_i^1)^{a_i^1}(x_i^2)^{a_i^2} y^{b_i}. \tag{14}$$

The budget constraint is

$$x_i^1 + p^2 x_i^2 + r_i C(p, y) = w_i^1 + p^2 w_i^2, \tag{15}$$

where we have used the normalized price $p = (p^1, p^2) \equiv (1, p^2)$. Here $C(p, y) = (2p^2 y)^{1/2}$. By Corollary 1, we know that the ratio equilibrium exists. In fact, we can find the specific equilibrium using Proposition 3 by finding the transformed Lindahl equilibrium. Using the transformation method, we have

$$u_i'(x_i^1, x_i^2, v^1, v^2) = (x_i^1)^{a_i^1}(x_i^2)^{a_i^2}(v^1)^{b_i}(v^2)^{b_i} \tag{16}$$

and the budget constraint now becomes

$$x_i^1 + p^2 x_i^2 + \gamma_i^1 v^1 + \gamma_i^2 v^2 = w_i^1 + p^2 w_i^2 \tag{17}$$

with $\sum_{i=1}^N \gamma_i = (1, p^2)$. It can be easily verified that the Lindahl equilibrium for the transformed economy $e'$, represented by (15) and (16), is unique. The private good's price $p^{*2}$, the Lindahl allocation $(x^*, v^*)$, and the personalized prices $\gamma_i^*$ are given by

$$p^{*2} = \frac{\sum_{i=1}^N [(1 - (a_i^1 + b_i)/c_i] w_i^1}{\sum_{i=1}^N w_i^2 (a_i^1 + b_i)/c_i}, \tag{18}$$

$$x_i^{*l} = \frac{a_i^l}{c_i} \frac{(w_i^1 + p^{*2} w_i^2)}{p^{*l}}, \qquad l = 1, 2, \tag{19}$$

$$v^{*l} = \sum_{i=1}^N \frac{b_i}{c_i} (w_i^1 + p^{*2} w_i^2)/p^{*l}, \qquad l = 1, 2, \tag{20}$$

$$\gamma_i^{*l} = \frac{b_i (w_i^1 + p^{*2} w_i^2)}{c_i v^{*l}}, \qquad l = 1, 2, \tag{21}$$

where $c_i = (a_i^1 + a_i^2 + 2b_i)$. By Proposition 3, we know the unique ratio equilibrium is $(x^*, y^*) \equiv (x^*, v^{*1} v^{*2})$ with $p^{*2}$ given by (18) and ratios $r_i^* = \gamma_i^{*1}/p^{*1} = \gamma_i^{*1}/1 = \gamma_i^{*1}$ for $i = 1, \ldots, N$ (cf. Remark 3).[12]

The second example reveals that although the differentiability of utility and production functions, in conjunction with the quasiconcavity of $u_i$, are sufficient conditions for Proposition 3, this assumption is not necessary for the conclusion of this proposition by showing that there exists a personalized price vector $\bar{\gamma}_i$ such that $\bar{\gamma}_{il}^* = r_{il}^* p^*$ with $\sum_{i=1}^N r_{il}^* = 1$.

[12] Note that we can also find the ratios from Eqs. (20), (21) from which $\gamma_i^{*2}/\gamma_i^{*1} = p^{*2}/p^{*1} = p^{*2}$. So $r_i^* = \gamma_i^{*2}/p^{*2} = \gamma_i^{*1}$.

EXAMPLE 2. For simplicity, we only consider the case of $J = 0$, $T = K = 1$, and $l = 2$.[13] Suppose the production function is $y \equiv y_s = \min\{(v^1)^\rho, (v^2/\delta)^\rho\}$, with $\rho \in R_{++}$ and utility functions $u_i(x_i^1, x_i^2, y) = (x_i^1)^{\alpha_i^1}(x_i^2)^{\alpha_i^2}y^{\beta_i}$, with $(\alpha_i^1, \alpha_i^2, \beta_i) \in R_{++}^3$ for all $i$. Then, for the transformed economy $e'$, we have

$$u_i'(x_i^1, x_i^2, v^1, v^2) = (x_i^1)^{\alpha_i^1}(x_i^2)^{\alpha_i^2}\left[\min\left\{(v^1)^\rho, \left(\frac{v^2}{\delta}\right)^\rho\right\}\right]^{\beta_i}$$

and the budget constraint is

$$x_i^1 + p^2x_i^2 + \gamma_i^1v^1 + \gamma_i^2v^2 = w_i^1 + p^2w_i^2.$$

We have again used the normalized price $p = (p^1, p^2) \equiv (1, p^2)$. It is clear that at the utility maximization point, we must have $v^1 = v^2/\delta$. Thus maximizing the transformed utility functions subject to the above budget constraints is equivalent to maximizing the following utility functions

$$u_i''(x_i^1, x_i^2, v^1) = (x_i^1)^{\alpha_i^1}(x_i^2)^{\alpha_i^2}(v^1)^{\rho\beta_i}$$

s.t.

$$x_i^1 + p^2x_i^2 + (\gamma_i^1 + \delta\gamma_i^2)v^1 = w_i^1 + p^2w_i^2.$$

Solving the Lindahl equilibrium for this transformed economy, we obtain $p^{*2} = (1/2a)\{-b + \sqrt{b^2 - 4ac}\}$, $\gamma_i^* + \delta\gamma_i^{*2} = [(\beta_iI_i^*/c_i)/\sum_{i=1}^N(\beta_iI_i^*/c_i)](1 + \delta p^{*2})$, $x_i^{*l} = (\alpha_i/c_ip^{*l})I_i^*$, $v^{*1} = \sum_{i=1}^N(\rho\beta_iI_i^*/c_i)(1/1 + \delta p^{*2})$, $v^{*2} = \delta v^{*1}$, where $a = \sum_{i=1}^N(\alpha_i^1/c_i)w_i^2$, $b = \sum_{i=1}^N w_i^2(1 - \alpha_i^2/c_i) - \sum_{i=1}^N w_i^1(1 - \alpha_i^1/c_i)$, $c = -\sum_{i=1}^N w_i^1(1 - \alpha_i^2/c_i)$, $I_i^* = (w_i^1 + p^{*2}w_i^2)$, and $c_i = \alpha_i^1 + \alpha_i^2 + \rho\beta_i$. Note that personalized prices $\gamma_i^{*1}$ and $\gamma_i^{*2}$ are not unique. However, if we let $\bar{\gamma}_i^{*1} = (\beta_iI_i^*/c_i)\sum_{i=1}^N(\beta_iI_{i=1}^*/c_i)$ and $\bar{\gamma}_i^{*2} = (\beta_iI_i^*/c_i)\sum_{i=1}^N(\beta_iI_i^*/c_i)p^{*2}$, then $\bar{\gamma}_i^{*1} + \delta\bar{\gamma}_i^{*2} = \gamma_i^{*1} + \delta\gamma_i^{*2}$. Thus by Proposition 3 and Remark 4, we know that the allocation $(x^*, y^*) \equiv (x^*, (v^{*1})^\rho)$ and the price system $(1, p^{*2}, r_i^*)$ with $r_i^* = \bar{\gamma}_i^{*1}$ is a ratio equilibrium.

Similarly, if utility functions $u_i$ are perfect complements, one can show that there exist Lindahl prices such that $\bar{\gamma}_{it}^* = r_i^{*\prime}p^*$. Thus the conclusion of Proposition 3 still holds.

[13] As mentioned earlier, Proposition 3 always holds without the differentiability and quasi-concavity assumptions for the case of one private good.

## 5. Concluding Remarks

In this paper, we define the ratio–Lindahl equilibrium for a mixed ownership economy. The ratio equilibrium is a special case of the ratio–Lindahl equilibrium; as a consequence, we generalize Kaneko's ratio equilibrium to a case with an arbitrary number of private goods. As mentioned earlier, a ratio–Lindahl equilibrium exists even when production technologies display some kinds of IRS. We prove the existence of ratio–Lindahl equilibria by proving the existence of Lindahl equilibria for the transformed private-ownership economy. This approach has a number of advantages: (1) it is simple in proving the existence; (2) it allows for the technologies of producing public goods by state-owned firms to display some kinds of IRS; and (3) it provides a simple way to find the ratio–Lindahl equilibrium by finding the conventional Lindahl equilibrium.

As is well known, the Lindahl mechanism is not "incentive-compatible" in the sense that it has the "free rider" problem. To solve this problem, many mechanisms have been designed in the literature which implement the Lindahl correspondence at Nash equilibrium points. See Hurwicz (1979b), Walker (1981), Tian (1989, 1990, 1991), and Tian and Li (1991), among others. The ratio–Lindahl mechanism, like the Lindahl mechanism, is not "incentive-compatible." Tian and Li (1994b) consider the incentive aspects of the ratio–Lindahl allocations by giving an "incentive-compatible" and "privacy preserving" mechanism whose Nash allocations coincide with ratio–Lindahl allocations for economies with one private good. By combining the techniques given by Tian (1992) and Tian and Li (1994b), one can also give a mechanism which implements ratio–Lindahl allocations for economies with any number of private goods.

## References


Berge, C. (1963). *Topological Spaces*, translated by E. M. Patterson. New York: Macmillan.

Foley, D. (1970). "Lindahl's Solution and the Core of an Economy with Public Goods," *Econometrica* **38**, 66–72.

Hurwicz, L. (1960). "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences* (K. J. Arrow, S. Karlin, and P. Suppes, Eds.). Stanford: Stanford Univ. Press.

Hurwicz, L. (1972a). "On Informationally Decentralized Systems," in *Decision and Organization* (R. Radner and C. B. McGuire, Eds.), volume in Honor of J. Marschak, pp. 297–336. Amsterdam: North-Holland.

Hurwicz, L. (1972b). "On the Dimension Requirements of Informationally Decentralized Pareto-Satisfactory Processes," in *The Process of Publication*, presented at the conference Seminar on Decentralization, Northwestern University, Evanston, IL.

Hurwicz, L. (1979a). "Socialism and Incentives: Developing a Framework," *J. Comparative Econ.* **3**, 207–216.

HURWICZ, L. (1979b). "Outcome Function Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Point," *Rev. Econ. Stud.* **46,** 217–225.

ITO, Y., AND KANEKO, M. (1981). "Linearization of Cost Functions in Public Good Economies," *Econ. Stud. Quart.* **32** (3), 237–246.

LANGE, O. (1936/1937). "On the Economic Theory of Socialism," *Rev. Econ. Stud.* **4.**

LERNER, A. P. (1944). *The Economics of Control.* New York.

KANEKO, M. (1977). "The Ratio Equilibrium and a Voting Game in a Public Goods Economy," *J. Econ. Theory* **16,** 123–136.

MAS-COLELL, A. (1974). "Continuous and Smooth Consumers: Approximation Theory," *J. Econ. Theory* **8,** 305–336.

MAS-COLELL, A. (1985). *The Theory of General Economic Equilibrium—A Differentiable Approach,* Cambridge: Cambridge Univ. Press.

MAS-COLELL, A., AND SILVESTRE, J. (1989). "Cost Share Equilibria: A Lindahl Approach," *J. Econ. Theory* **47,** 239–256.

MAS-COLELL, A., AND SILVESTRE, J. (1991). "A Note on Cost-Share Equilibrium and Owner-Consumers," *J. Econ. Theory* **54,** 204–214.

MILLERON, J.-C. (1972). "Theory of Value with Public Goods: A Survey Article," *J. Econ. Theory* **5,** 419–477.

ROBERTS, D. J. (1974). "The Lindahl Solution for Economies with Public Goods," *J. Public Econ.* **3,** 23–42.

SAIJO, T. (1991). "Incentive Compatibility and Individual Rationality in Public Good Economies," *J. Econ. Theory* **55,** 203–212.

TIAN, G. (1989). "Implementation of the Lindahl Correspondence by a Single-Valued, Feasible, and Continuous Mechanism," *Rev. Econ. Stud.* **56,** 613–621.

TIAN, G. (1990). "Completely Feasible and Continuous Nash-Implementation of Lindahl Correspondence with a Message Space of Minimal Dimension," *J. Econ. Theory* **51,** 443–452.

TIAN, G. (1991). "Implementation of Lindahl Allocations with Nontotal-Nontransitive Preferences," *J. Public Econ.* **46,** 247–259.

TIAN, G. (1992). "Implementation of the Walrasian Correspondence without Continuous, Convex, and Ordered Preferences," *Social Choice Welfare* **9,** 117–130.

TIAN, G. (1994). "On Incentive and Information Aspects of Resource Allocating Mechanisms in Economies with Public Goods," *J. Math. Econ.*, forthcoming.

TIAN, G., AND LI, Q. (1991). "Completely Feasible and Continuous Implementation of the Lindahl Correspondence With Any Number of Goods," *Math. Soc. Sci.* **21,** 67–79.

TIAN, G., AND LI, Q. (1994). "An Implementable and Informationally Efficient State-Ownership System with General Variable Returns," *J. Econ. Theory,* forthcoming.

TIAN, G., AND LI, Q. (1990b). "Ratio-Lindahl Equilibria and an Informationally Efficient and Implementable Mixed-Ownership System," *J. Econ. Behavior and Organization,* forthcoming.

VON HAYEK, F. A. (1935). "The Present State of the Debate," in *Collectivist Economic Planning* (F. A. von Hayek, Ed.). London.

VON HAYEK, F. A. (1945). "The Use of Knowledge in Society," *Amer. Econ. Rev.* **35,** 519–530.

WALKER, M. (1981). "A Simple Incentive Compatible Mechanism for Attaining Lindahl Allocations," *Econometrica* **49,** 65–71.

WEBER, S., AND WIESMETH, H. (1991). "The Equivalence of Core and Cost Share Equilibria in an Economy with a Public Good," *J. Econ. Theory* **54,** 180–197.