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CENTRAL LIMIT THEOREM FOR DEGENERATE *U*-STATISTICS OF ABSOLUTELY REGULAR PROCESSES WITH APPLICATIONS TO MODEL SPECIFICATION TESTING

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Under quite general conditions we establish a central limit theorem for second order degenerate *U*-statistics of absolutely regular processes. The new central limit theorem is then used to establish the validity of an asymptotic test for the parametric functional form of a general regression model involving time series.

Keywords: Degenerate *U*-Statistics; central limit theorem; absolutely regular process; model specification test

1. INTRODUCTION

Since Hoeffding (1948) introduced the concept of a *U*-statistic, Central Limit Theorems (CLTs) for *U*-statistics have played an important role in deriving aysmptotic properties of both estimators and test statistics, the reason being that most of the estimators and test statistics either can be written as *U*-statistics or contain *U*-statistics as components. See Lee (1990) for a review of some basic results on *U*-statistics.

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A second order U-statistic of a sample of n observations $\{Z_t\}_{t=1}^n$ takes the following form:

$$U_n = \binom{n}{2}^{-1} \sum_{1 \le s < t \le n} H_n(Z_t, Z_s),$$

where the 'kernel function' H_n may depend on n. The asymptotic distribution of U_n depends on whether its kernel H_n depends on n or not (constant kernel *versus* variable kernel) and whether it is degenerate or non-degenerate.[1]

Recently, many nonparametric estimators such as kernel and series estimators have been used to estimate functionals of unknown density or regression functions, or form model specification tests. This leads to U-statistics with variable kernels. Examples, can be found in Powell *et al.* (1989); Härdle and Stoker (1989); Robinson (1989); Stoker (1989) and numerous papers on consistent model specification tests using nonparametric estimators (see Fan and Li, 1996a for references). Derivation of asymptotic distributions of these estimators or test statistics requires CLTs for U-statistics with variable kernels. For non-degenerate case, Powell *et al.* (1989) present a result for second order U-statistics when the observations are independent. For degenerate U-statistics with variable kernels and independent observations, Hall (1984) showed that under appropriate conditions, a second order degenerate U-statistic has an asymptotic normal distribution; Fan and Li (1996a) extended Hall's result to higher order case; De Jong (1987) and Khashimov (1988) also presented CLTs for degenerate U-statistics.

The CLTs in Hall (1984); De Jong (1987) and Fan and Li (1996a) for independent observations have proved to be an indispensable tool in establishing consistent model specification tests, see Fan and Li (1996a) for references. However, the corresponding CLTs for dependent observations are not well established. This has prevented similar tests to be developed for time series models, see *e.g.*, Tjostheim (1994) and Hjellvik and Tjostheim (1995). The papers by Khashimov (1987, 1992); Takahata and Yoshihara (1987) and Yoshihara (1989, 1992)

[1]Fan and Li (1996b) provide a detailed discussion on the existing asympotic results for U-statistics with constant kernels and the applications of these results in hypothesis testing. We note that the asymptotic null distributions of the tests in Bierens (1982); Bierens and Ploberger (1997) and Linton and Gozalo (1995) may also be derived by using results on degenerate U-statistics with constant kernels.

present CLTs for degenerate U-statistics with variable kernels for dependent observations. Khashimov (1987) established CLTs for degenerate U_n of m-dependent processes. Khashimov (1992) and Yoshihara (1989, 1992) considered degenerate U-statistics of absolutely regular processes and/or other weakly dependent processes. Takahata and Yoshihara (1987) proved the asymptotic normality of the integrated squared error of the kernel density estimator for absolutely regular processes. Apparently, m-dependence is too strong a requirement; Takahata and Yoshihara (1987) did not provide a general result; Although Khashimov (1992) provided a CLT for second order degenerate U-statistics, one of his conditions when applied to testing the parametric functional form of a regression function requires that the error term be bounded. See Fan and Li (1997) for details. The conditions in Yoshihara (1989, 1992) are not directly imposed on the kernel function of the U-statistic and hence are difficult to check in practice.

The purpose of this paper is to provide a general CLT for second order degenerate U-statistics with variable kernels for absolutely regular processes. This is accomplished in Section 2. The new CLT can be used to relax the boundedness of the error term when applied to consistent model specification tests in the context of a regression model. We demonstrate this in Section 3. Appendix A contains the proof of Theorem 2.1.

2. CLT FOR DEGENERATE U-STATISTICS

Let $\{Z_t\}$ be a strictly stationary stochastic process and $\mathcal{M}_s^t(Z)$ denote $\sigma(Z_s,\ldots,Z_t)$, the sigma algebra generated by $(Z_s,\ldots,Z_t)$ for $s \leq t$. The process $\{Z_t\}$ is called absolutely regular, if as $\tau \to \infty$, $\beta_\tau = \sup_{s\in N} E[\sup_{A\in\mathcal{M}_{s+\tau}^{\infty}(Z)}\{|P(A|\mathcal{M}_{-\infty}^{s}(Z)) - P(A)|\}] \to 0$.

For convenience, we write

$$U_n = \sum \sum_{1\leq s<t\leq n} H_n(Z_t, Z_s),$$

where H_n depends on n and satisfies $\int H_n(x,y)dF(x) = 0$ for all y, and $F(\cdot)$ is the marginal distribution function of $\{Z_t\}$.

In this section, we will establish a CLT for U_n when $\{Z_t\}$ is an absolutely regular process. Without loss of generality, we assume $E(U_n) = 0$.

Also, we will suppress the dependence of $H_n(\cdot,\cdot)$ on n and write $H(x,y) = H_n(x,y)$.

Our method of proof is the same as that of Takahata and Yoshihara (1987). That is, we first decompose the double sum in U_n into sums of large and small blocks; Then we verify the CLT for the sum of large blocks by using Theorem 2.2 in Dvoretzky (1972) and show that the sum of small blocks does not affect the asymptotic distribution of U_n.

We now introduce some notations. Let $F_{i_1,\ldots,i_j}$ denote the joint distribution function of $\{Z_{i_1},\ldots,Z_{i_j}\}$, $j = 2, 3, 4$; $dQ_{i_1,i_2}(z_{i_1}, z_{i_2})$ denote either $dF(z_{i_1})dF(z_{i_2})$, or $dF_{i_1,i_2}(z_{i_1}, z_{i_2})$; $dQ_{i_1,i_2,i_3}(z_{i_1}, z_{i_2}, z_{i_3})$ be either $dF(z_{i_1})dF(z_{i_2})dF(z_{i_3})$, or $dF_{i_1,i_2,i_3}(z_{i_1}, z_{i_2}, z_{i_3})$ or $dF(z_{j_1})dF_{j_2,j_3}(z_{j_2}, z_{j_3})$, where $\{j_1, j_2, j_3\}$ is any possible permutation of $\{i_1, i_2, i_3\}$; $dQ_{i_1,i_2,i_3,i_4}(z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4})$ is similarly defined. Also let $\{\tilde{Z}_t\}_{t=1}^n$ be an i.i.d. sequence having the same marginal distribution as $\{Z_t\}$ and define

$$G(x,y) = E[H(Z_1,x)H(Z_1,y)];\ \sigma_n^2 = E[H^2(\tilde{Z}_1,\tilde{Z}_2)];\ \mu_{n_4} = E[H^4(\tilde{Z}_1,\tilde{Z}_2)];$$

$$\gamma_{nij} = \max_{t\neq s,\ t'\neq s'} E[H^i(Z_t,Z_s)H^j(Z_{t'},Z_{s'})],\ (i,j) = (1,1), (2,2) \text{ or } (1,3);$$

$$\tilde{\gamma}_{n14} = \max_{t\neq s} \int \{E[H(z,Z_t)H(z,Z_s)]\}^2 dF(z);$$

$$\tilde{\gamma}_{n22} = E[H^2(\tilde{Z}_1,\tilde{Z}_2)H^2(\tilde{Z}_1,\tilde{Z}_3)];$$

$$\gamma_n = \max\{\gamma_{n11}, \tilde{\gamma}_{n22}, \tilde{\gamma}_{n14}\};\quad \nu_n = \max\{\gamma_{n22}, \gamma_{n13}\};$$

$$\sigma_G^2 = E[G^2(Z_t,Z_t)];\quad \mu_{nG2} = \max_{t\neq s} \int G^2(Z_t,Z_s)dQ(Z_t,Z_s);$$

$$\begin{aligned}\gamma_{nG11} = \max\Big\{ &\max_{s\neq s'\neq s''} |E[G(Z_s,Z_s)G(Z_{s'},Z_{s''})]|,\\ &\max_{s\neq s'\neq s''} |E[G(Z_s,Z_{s'})G(Z_s,Z_{s''})]|,\\ &\max_{s\neq s'\neq s''\neq s'''} |E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})]|\Big\};\end{aligned}$$

$$M_{nij} = \max_{s\neq t,\ s'\neq t'} \int |H^i(z_t,z_s)H^j(z_{t'},z_{s'})|^{1+\delta} dQ_{t,s,t',s'}(z_t,z_s,z_{t'},z_{s'}),$$

i, j are non-negative integers;

$$M_{nG1} = \max_{t\neq s} \int |G(Z_t,Z_s)|^{1+\delta} dQ(Z_t,Z_s);$$

$$M_{nG11} = \max_{s,s',s'',s'''} \int |G(z_s,z_{s'})G(z_{s''},z_{s'''})|^{1+\delta} dQ_{s,s',s'',s'''}(z_s,z_{s'},z_{s''},z_{s'''});$$

$$M_n = \max\{M_{n10}, M_{n11}, M_{n40}, M_{n22}, M_{n13}, M_{nG1}, M_{nG11}\}.$$

To construct small and large blocks, we let $r = r_n = [n^{1/4}]$, $m = m_n = o(r)$ ($m \geq 1$, usually $m \to \infty$ as $n \to \infty$), and $k = k_n = [n/(r+m)]$, where r and m are respectively the number of elements in the large and small blocks. Define a sequence $\{(a_i, b_i)\,(i = 1, \ldots, k)\}$ as follows:

$$b_0 = 0, \;\; a_i = b_{i-1} + m, \;\; b_i = a_i + r - 1, \quad (i = 1, \ldots, k).$$

Then we can decompose U_n in the following way:

$$U_n = V_n + B_n + Q_n,$$

where

$$V_n = \sum_{\alpha=1}^{k} \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha - m+1} H(Z_t, Z_s), \quad B_n = \sum_{\alpha=1}^{k} \sum_{t=a_\alpha}^{b_\alpha} \sum_{a_\alpha - m+2 \leq s < t} H(Z_t, Z_s),$$

$$Q_n = \sum_{\alpha=1}^{k} \sum_{t=b_\alpha+1}^{a_{\alpha+1}-1} \sum_{s<t} H(Z_t, Z_s).$$

We will establish a CLT for U_n by showing that $\sqrt{2}V_n/(n\sigma_n) \to N(0,1)$ in distribution, $B_n/(n\sigma_n) = o_p(1)$, and $Q_n/(n\sigma_n) = o_p(1)$. To verify these results, we adopt the following assumptions: As $n \to \infty$,

(A1) (i) $\mu_{n4} r^2 m/(n^2\sigma_n^4) = o(1)$, (ii) $\gamma_n m^2/\sigma_n^2 = o(1)$, (iii) $\nu_n r m^5/(n^2\sigma_n^4) = o(1)$;
(A2) (i) $\mu_{nG2} m^4/\sigma_n^4 = o(1)$, (ii) $\gamma_{nG11} m^4/\sigma_n^4 = o(1)$, (iii) $\sigma_G^2 m/(n\sigma_n^4) = o(1)$;
(A3) (i) $n^2 \beta_m^{\delta/(1+\delta)}(m^2 + n^2\beta_m^{\delta/(1+\delta)})/\sigma_n^4 = o(1)$, (ii) $M_n = O(1)$.

If $\{Z_t\}$ is an i.i.d. process, one can take $m = 1$ in the above conditions. Then (A2) (i) becomes $E[G^2(\tilde{Z}_1, \tilde{Z}_2)]/\sigma_n^4 = o(1)$, and it is easy to see that the rest of the above conditions either hold trivially or are implied by $E[H^4(\tilde{Z}_1, \tilde{Z}_2)]/(n\sigma_n^4) = o(1)$. Thus we get

$$\frac{E[G^2(\tilde{Z}_1, \tilde{Z}_2)] + n^{-1}E[H^4(\tilde{Z}_1, \tilde{Z}_2)]}{\sigma_n^4} \to 0,$$

the condition used in Hall (1984) for i.i.d. data case.

With the above conditions, we now state the main result of this paper.

THEOREM 2.1 *Let $\{Z_t\}$ be a strictly stationary, absolutely regular process, and assume that Assumptions* ($A1$) *to* ($A3$) *hold. Then*

$$\frac{\sqrt{2}U_n}{n\sigma_n} \to N(0,1) \quad \textit{in distribution.}$$

A detailed proof of Theorem 2.1 is given in Appendix A.

We will apply Theorem 2.1 to validating an asymptotic test for the parametric functional form of a regression model involving time series in the next section.

3. A SIMPLE TEST FOR REGRESSION FUNCTIONAL FORM

Let $Z_t = (Y_t, X_t')'$ be an absolutely regular process, where Y_t is a scalar, and X_t is $p \times 1$ which may contain lagged values of Y_t. Often interest is in testing if $E(Y_t|X_t)$ belongs to a specific parametric family. This can be characterized by the null hypothesis of the following form:

$$H_0: P[E(Y_t|X_t) = g(X_t, \gamma_0)] = 1, \quad \text{for some } \gamma_0 \in \mathcal{B} \subset R^q.$$

We take the alternative hypothesis as

$$H_1: P[E(Y_t|X_t) = g(X_t, \gamma)] < 1, \quad \text{for all } \gamma \in \mathcal{B} \subset R^q.$$

Let $\varepsilon_t = Y_t - g(X_t, \gamma_0)$. Then under H_0, $E(\varepsilon_t|X_t) = 0$ almost everywhere. For i.i.d. observations, Li and Wang (1996) and Zheng (1996) established a consistent test for H_0 *versus* H_1 based on a consistent kernel estimator of $E[\varepsilon_t E(\varepsilon_t|X_t) f(X_t)]$ defined as

$$I_n = \frac{1}{n^2 h^p} \sum \sum_{s \neq t} \hat{\varepsilon}_t \hat{\varepsilon}_s K_{st}, \tag{1}$$

where $f(\cdot)$ is the probability density function of X_t, $\hat{\varepsilon}_t = Y_t - g(X_t, \hat{\gamma})$, $\hat{\gamma}$ is the nonlinear least squares (NLS) estimator of γ_0 under H_0,

$h \equiv h_n \to 0$ is a sequence of smoothing parameters, $K_{st} = K((X_s - X_t)/h)$, and $K(\cdot)$ is a kernel function satisfying certain conditions.

By using the CLT in Hall (1984) for i.i.d. observations, Li and Wang (1996) and Zheng (1996) showed that under H_0, I_n is asymptotically normally distributed. Fan and Li (1997) extended this test to the case where $X_t = Y_{t-1}$ and g is linear. They established the asymptotic normality of I_n under H_0 for absolutely regular process $\{Y_t\}$ by using the CLT in Khashimov (1992). However, one of the conditions in Khashimov (1992) requires that the error term ε_t be bounded. In this section, we'll show that the boundedness of ε_t can be relaxed by using Theorem 2.1 developed in Section 2. In addition, we allow X_t to contain a finite number of lagged values of Y_t and other exogenous variables, and allow the function g to be nonlinear.

First, we list some assumptions:

(C1) (i) With probability one, $E[\varepsilon_t|\mathcal{M}_{-\infty}^{t}(X), \mathcal{M}_{-\infty}^{t-1}(Y)] = 0$; (ii) $E[|\varepsilon_t|^{4+\eta}] < \infty$ and $E[|\varepsilon_{t_1}^{i_1}\varepsilon_{t_2}^{i_2}, \ldots, \varepsilon_{t_l}^{i_l}|^{\xi}] < M < \infty$, where η is an arbitrarily small positive number, ξ is slightly larger than one, $2 \leq l \leq 4$ is an integer, $0 \leq i_j \leq 4$ and $\sum_{j=1}^{l} i_j = 8$; (iii) Let $\sigma^2(x) = E(\varepsilon_t^2|X_t = x)$ and $\mu_4(x) = E(\varepsilon_t^4|X_t = x)$. Then $\sigma^2(x)$ and $\mu_4(x)$ satisfy some Lipschitz conditions: $|\sigma^2(x+u) - \sigma^2(x)| \leq D(x)\,\|u\|$ and $|\mu_4(x+u) - \mu_4(x)| \leq D(x)\|u\|$ with $E[|D(X)|^{2+\eta'}] < \infty$ for some $\eta' > 0$.

(C2) (i) $\nabla g(X, \cdot)$ and $\nabla^2 g(X, \cdot)$ are continuous in X and dominated by a function (say $M_g(X)$) with finite second moments, where $\nabla g(X, \cdot)$ and $\nabla^2 g(X, \cdot)$ are $q \times 1$ vector of first order partial derivatives and $q \times q$ matrix of second order partial derivatives of g with respect to γ respectively; (ii) $E[\nabla g(X, \beta)\nabla' g(X, \beta)]$ is nonsingular for γ in a neighborhood of *plim* $\hat{\gamma}$; (iii) Let $f_{\tau_1,\ldots,\tau_l}(.,\ldots,.)$ be the joint probability density function of $(X_1, X_{1+\tau_1}, \ldots, X_{1+\tau_l})$ $(1 \leq l \leq 3)$. Then for all $(\tau_1, \ldots, \tau_l)$, $f_{\tau_1,\ldots,\tau_l}(.,\ldots,.)$ exists and satisfies a Lipschitz condition: $|f_{\tau_1,\ldots,\tau_l}(x_1 + u_1, x_2 + u_2, \ldots, x_l + u_l) - f_{\tau_1,\ldots,\tau_l}(x_1, x_2, \ldots, x_l)| \leq D_{\tau_1,\ldots,\tau_l}(x_1, x_2, \ldots, x_l)\|u\|$, where $D_{\tau_1,\ldots,\tau_l}(.,\ldots,.)$ is integrable and satisfies the conditions that $\int D_{\tau_1,\ldots,\tau_l}(x, x, \ldots, x)M_g^{2\xi}(x)dx < M < \infty$, $\int f_{\tau_1,\ldots,\tau_l}(x, x, \ldots, x)M_g^{2\xi}(x)dx < M < \infty$, and $\int D_{\tau_1,\ldots,\tau_l}(x_1, x_2, \ldots, x_l)f_{\tau_1,\ldots,\tau_l}(x_1, x_2, \ldots, x_l)dx < M < \infty$ for some $\xi > 1$.

(C3) $K(\cdot)$ is bounded and symmetric with $\int K(u)\,du = 1$ and $\int \|u\|^2 K(u)\,du < \infty$.

(C4) The smoothing parameter $h = O(n^{-\bar{\alpha}})$ for some $0 < \bar{\alpha} < (7/8)p$.

(C5) The process $\{Z_t\}$ is absolutely regular with mixing coefficient $\beta_\tau = O(\rho^\tau)$ $(0 < \rho < 1)$.

(C6) $\hat{\gamma} - \bar{\gamma} = o_p(1)$ under H_1 for some $\bar{\gamma} \in R^q$.

We now briefly comment on the above assumptions. (C1) (i) states that the innovations $\{\varepsilon_t\}$ is a margingal difference; (C1) (ii) imposes some moment conditions on $\{\varepsilon_t\}$: For i.i.d. observations, these conditions are equivalent to $E[|\varepsilon_t|^{4+\eta}] < \infty$ for some arbitrarily small positive number η; (C1) (iii) contains some smoothness conditions on the second and fourth conditional moment functions of ε_t. (C2) (i) and (ii) are standard assumptions adopted in nonlinear regression models. In particular, they ensure that under H_0 and (C5), $\hat{\gamma} - \gamma_0 = O_p(n^{-1/2})$. (C2) (iii) contains some Lipschitz type conditions and moment conditions. (C3) is a standard assumption on the kernel function; (C4) implies that $(\log n)h^{\eta_1} \to 0$ and $n^{7/8}h^p/(\log n)^{\eta_2} \to \infty$ for arbitrary positive constants η_1 and η_2. It allows the choice of a wide range of smoothing parameter values and is slightly stronger than the usual conditions of $h \to 0$ and $nh^p \to \infty$. (C5) requires that the underlying process $\{Z_t\}$ be absolutely regular with geometric decay rate. This is not a very restrictive assumption, because many well-known processes satisfy (C5), see the examples in Fan and Li (1997). (C6) is known to hold under general conditions, see *e.g.*, Fuller (1996) and White (1994).

The next result concerns the asymptotic distribution of I_n under H_0.

THEOREM 3.1 *Under Assumptions (C1)–(C5) and H_0, we have $nh^{p/2}I_n/\hat{\sigma}_0 \to N(0,1)$ in distribution, where $\hat{\sigma}_0^2 = (1/n^2h^p)\sum_t\sum_{s\neq t}\hat{\varepsilon}_t^2\hat{\varepsilon}_s^2 K_{ts}^2$ is a consistent estimator of $\sigma_0^2 = 2\int K^2(u)du \int f^2(x)\sigma^4(x)dx$ with $\sigma^2(x) = E(\varepsilon_t^2|X_t = x)$.*

Proof Noticing that under H_0, $\hat{\varepsilon}_t = \varepsilon_t - [g(X_t,\hat{\gamma}) - g(X_t,\gamma_0)]$, we can rewrite I_n in the following way:

$$
\begin{aligned}
I_n &= \frac{1}{n^2h^p}\sum\sum_{t\neq s}\varepsilon_t\varepsilon_s K_{st} - \frac{2}{n^2h^p}\sum\sum_{t\neq s}[g(X_s,\hat{\gamma}) - g(X_s,\gamma_0)]\varepsilon_t K_{st} \\
&\quad + \frac{1}{n^2h^p}\sum\sum_{t\neq s}[g(X_t,\hat{\gamma}) - g(X_t,\gamma_0)][g(X_s,\hat{\gamma}) - g(X_s,\gamma_0)]K_{st} \\
&\equiv I_{n1} - 2I_{2n} + I_{3n}. \qquad (2)
\end{aligned}
$$

We will prove Theorem 3.1 by showing that (i) $nh^{p/2}I_{1n} \to N(0, \sigma_0^2)$ in distribution, (ii) $\hat{\sigma}_0^2 = \sigma_0^2 + o_p(1)$; (iii) $I_{2n} = o_p((nh^{p/2})^{-1})$; and (iv) $I_{3n} = O_p(n^{-1})$.

Proof of (i): $nh^{p/2}I_{1n} \to N(0, \sigma_0^2)$ in distribution.

To apply Theorem 2.1, we let $U_n = \sum\sum_{1 \le s < t \le n} \varepsilon_t \varepsilon_s K_{ts} \equiv \sum\sum_{1 \le s < t \le n} H(Z_t, Z_s)$, where $Z_t = (X_t', \varepsilon_t)'$. We now verify Conditions (A1) to (A3) of Theorem 2.1 under (C1) to (C6). Let r, m, a_i and b_i $(i = 1, \ldots, k)$ be defined as in Section 2 with $m = [C \log n]$, where C is a (large) positive constant.

We will use the notation $\bar{a}_n \sim \bar{b}_n$ to denote that $\bar{a}_n$ and $\bar{b}_n$ have the same order of magnitude. First we check (A1). Obviously $\sigma_n^2 = E[H^2(\tilde{Z}_1, \tilde{Z}_2)] = \int\int \sigma^2(x)\sigma^2(y)K^2(x - y/h)f(x)f(y)dxdy = O(h^p)$ and $\mu_{n4} = E[H^4(\tilde{Z}_1, \tilde{Z}_2)] = \int\int \mu_4(x)\mu_4(y)K^2(x-y/h)f(x)f(y)dxdy = O(h^p)$.

Denote $K_{x,s_l} = K((x - X_{s_l})/h)$ $(l = 1, \ldots, 4)$, then by (C2) (iii), we have

$$\begin{aligned}\tilde{\gamma}_{n14} &= \int \mu_4(x) E[\varepsilon_{s_1}\varepsilon_{s_2}K_{x,s_1}K_{x,s_2}]E[\varepsilon_{s_1}\varepsilon_{s_2}K_{x,s_3}K_{x,s_4}]\, f(x)dx \\ &\le \int \mu_4(x)\{E[\varepsilon_{s_1}^2\varepsilon_{s_2}^2]E[K_{x,s_1}^2K_{x,s_2}^2]E[\varepsilon_{s_3}^2\varepsilon_{s_4}^2]E[K_{x,s_3}^2K_{x,s_4}^2]\}^{1/2}f(x)dx \\ &\le \int \mu_4(x)\{Ch^{4p}\}^{1/2}f(x)dx = O(h^{2p}).\end{aligned}$$

$$\begin{aligned}\tilde{\gamma}_{n22} = \int\int\int \mu_4(x)\sigma^2(y)\sigma^2(z)K^2((x - y)/h)K^2((x - z)/h) \\ \times f(x)f(y)f(z)dxdydz = O(h^{2p}).\end{aligned}$$

$\gamma_{n11} \sim |E[H(Z_t, Z_s)H(Z_{t'}, Z_{s'})]| = |E\,|\varepsilon_t\varepsilon_s\varepsilon_{t'}\varepsilon_{s'}K_{ts}K_{t's'}]| \le [E|\varepsilon_t\varepsilon_s\varepsilon_{t'}\varepsilon_{s'}|^{\xi}]^{1/\xi}[E|K_{ts}K_{t's'}|^{\eta}]^{1/\eta} = O(h^{2p/\eta})$ by assumptions (C1) and (C2) (iii), where ξ is slightly larger than 2 and $\eta = (1 - \xi^{-1})^{-1}$. Hence, $\gamma_n = O(h^{2p/\eta})$ $(1 < \eta < 2)$.

Similarly one can show that $\gamma_{n13} \sim \gamma_{n22} \le [E|\varepsilon_t^2\varepsilon_{t'}^2\varepsilon_s^2\varepsilon_{s'}^2|^{\xi'}]^{1/\xi'}[E|K_{ts}^2K_{t's'}^2|^{\eta'}]^{1/\eta'} = O(h^{2p/\eta'})$, where ξ' is slightly larger than 1 and $\eta' = (1 - (1/\xi'))^{-1} > 1$. Hence, $\nu_n = O(h^{2p/\eta'})$.

Summarizing the above, we have shown that $\sigma_n^2 = O(h^p)$, $\mu_{n4} = O(h^p)$, $\gamma_n = O(h^{2p/\eta})$, $\nu_n = O(h^{2p/\eta'})$, where $1 < \eta < 2$, and $\eta' > 1$. These results, together with (C4), imply (A1) (i)–(iii).

Next, consider

$$G(z_t,z_s)=E[H(Z,z_t)H(Z,z_s)]=\varepsilon_t\varepsilon_s\int\sigma^2(x)K\left(\frac{x-x_t}{h}\right)K\left(\frac{x-x_s}{h}\right)f(x)dx$$
$$=h^p\varepsilon_t\varepsilon_s\int\sigma^2(x_t+hv)K(v)K\left(\frac{x_t-x_s}{h}+v\right)f(x_t+hv)dv. \qquad (3)$$

Noting that $G(z_t,z_s)$ given in (3) already has a h^p factor, and then using the same arguments as above (by Hölder's inequality), it is straightforward to show that

$$\sigma_G^2=E[G^2(Z_t,Z_t)]=O(h^{2p}),\quad \mu_{nG2}\sim E[G^2(Z_t,Z_s)]=O(h^{2p}h^{p/\eta'})$$
$$\text{and}\quad \gamma_{nG11}=O(h^{2p}h^{p/\eta'}),$$

where $\eta'>1$ (η' can be very large). Thus (A2) (i)–(iii) are satisfied.

Finally it is easy to show that M_n is bounded by some positive constant given (C1)–(C3). Also $\sigma_n^2=O(h^p)$ and $\beta_m=O(\rho^m)=O(\rho^{-C\gamma\log_\rho n})=O(n^{-C\gamma})$, where $\gamma=-\log\rho>0$. Hence we have $m^2n^2\beta_m^{\delta/(1+\delta)}/\sigma_n^4=o(1)$ which implies $n^2\beta_m^{\delta/(1+\delta)}=o(1)$ provided we choose C sufficiently large. Thus (A3) (i)–(ii) are all satisfied.

Hence, by the result of Theorem 2.1 and $E(U_n)=0$ (by (C1)), we get $\sqrt{2}U_n/(n\sigma_n)\to N(0,1)$ in distribution. By noting that $nh^{p/2}I_{1n}=(2/(n-1)h^{p/2})U_n$ and using the fact that $\sigma_n=h^{p/2}[(\sigma_0\sqrt{2})+o(1)]$, we obtain $nh^{p/2}I_{1n}\to N(0,\sigma_0^2)$ in distribution. This finishes the proof of (i).

Proof of (ii): $\hat{\sigma}_0^2=\sigma_0^2+o_p(1)$.

Under the assumptions of Theorem 3.1, it is easy to show that

$$\hat{\sigma}_0^2=\frac{2}{n(n-1)h^p}\sum\sum_{t\neq s}\varepsilon_t^2\varepsilon_s^2K_{ts}^2+o_p(1)\equiv\bar{\sigma}_0^2+o_p(1),$$

and $\mathrm{Var}(\bar{\sigma}_0^2)=o(1)$. Hence in order to prove (ii), it suffices to show that $\bar{\sigma}_0^2=E[\bar{\sigma}_0^2]+o_p(1)$. By Lemma 1 in Yoshihara (1976), we have $E[\bar{\sigma}_0^2]-\sigma_0^2=(nh^p)^{-1}O\left(\sum_{\tau=1}^n\beta_\tau\right)=O((nh^p)^{-1})=o(1)$.

Proof of (iii): $I_{2n} = o_p((nh^{p/2})^{-1})$.

Using $g(X_t, \hat{\gamma}) - g(X_t, \gamma_0) = \nabla' g(X_t, \gamma_0)(\hat{\gamma} - \gamma_0) + 1/2(\hat{\gamma} - \gamma_0)' \nabla^2 g(X_t, \tilde{\gamma})(\hat{\gamma} - \gamma_0)$, where $\tilde{\gamma}$ is between $\hat{\gamma}$ and γ_0, we get

$$
\begin{aligned}
I_{2n} = \frac{1}{n^2 h^p} \sum\sum_{t \neq s} \{ & (\hat{\gamma} - \gamma_0)' \varepsilon_s \nabla g(X_t, \gamma_0) \\
& + (\hat{\gamma} - \gamma_0)' \varepsilon_s \nabla^2 g(X_t, \hat{\gamma})(\hat{\gamma} - \gamma_0)/2\} K_{ts} \\
\equiv (\hat{\gamma} - \gamma_0)' A_{1n} & + (\hat{\gamma} - \gamma_0)' A_{2n} (\hat{\gamma} - \gamma_0).
\end{aligned}
$$

We first consider A_{1n}. Let (a) denote the case of $\min\{|s - s'|, |s - t|, |s - t'|\} > m$, (b) the case of $\min\{|s - s'|, |s - t|, |s - t'|\} \leq m$ and $W_t \equiv \nabla g(X_t, \gamma_0)$. We have

$$
\begin{aligned}
E(\|A_{1n}\|^2) &= (n^2 h^p)^{-2} \sum_t \sum_{s \neq t} \sum_{t'} \sum_{s' \neq t'} E[W_t' \varepsilon_s W_{t'} \varepsilon_{s'} K_{ts} K_{t's'}] \\
&= (n^2 h^p)^{-2} \left\{ \sum\nolimits_{(a)} + \sum\nolimits_{(b)} \right\} E[W_t' \varepsilon_s W_{t'} \varepsilon_{s'} K_{ts} K_{t's'}] \\
&\leq (n^2 h^p)^{-2} \{ c n^4 \beta_m^{\delta/(1+\delta)} + m n^3 \} \max_{t \neq s, t' \neq s'} E[W_t' W_{t'} \varepsilon_s \varepsilon_{s'} K_{ts} K_{t's'}] \\
&= (n^2 h^p)^{-2} \{ o(1) + n^3 O(h^{2p/\eta}) \}
\end{aligned}
$$

for some $1 < \eta < 2$, because we can choose $C > 4(1 + \delta)/(\gamma\delta)$ in $m = [C \log(n)]$ and

$$
\begin{aligned}
& \max_{t \neq s, t' \neq s'} E[W_t' W_{t'} \varepsilon_s \varepsilon_{s'} K_{ts} K_{t's'}] \\
& \quad \leq c \max_{t \neq s, t' \neq s'} E[M(X_t) M(X_{t'}) \varepsilon_s \varepsilon_{s'} K_{ts} K_{t's'}] \\
& \quad \leq c \max_{t \neq s \neq t'} \{ E[|\varepsilon_s \varepsilon_{s'}|^{\xi}] \}^{1/\xi} \{ E[(M(X_t) M(X_{t'}) K_{ts} K_{t's'})^{\eta}] \}^{1/\eta} \\
& \quad = O(h^{2p/\eta})
\end{aligned}
$$

by (C2) (iii), where $\eta = (1 - \xi^{-1})^{-1}$ ($\xi > 2$, $1 < \eta < 2$).

Hence $E\|A_{1n}\|^2 = o((n^2 h^p)^{-2}) + O(m(nh^{2p(\eta-1)/\eta})^{-1})$ and this implies $A_{1n} = o_p((n^2 h^p)^{-1}) + O_p(m^{1/2}(n^{-1/2} h^{-p(\eta-1)/\eta}))$, which leads to $A_{1n}(\hat{\gamma} - \gamma) = n^{-1/2} o_p((n^2 h^p)^{-1}) + O_p(m^{1/2} n^{-1} h^{-p(\eta-1)/\eta}) = o_p((nh^{p/2})^{-1})$ because $1 < \eta < 2$, $m = [C \log(n)]$ and $h \sim n^{-\bar{\alpha}}((7/8)p > \bar{\alpha} > 0)$.

It remains to evaluate the order of A_{2n}. By (C2) (i), we have

$$\begin{aligned}
E\|A_{2n}\| &\leq (n^2h^p)^{-1}n^2 \max_{t\neq s} E\|\varepsilon_s M(X_t)K_{ts}\| \\
&\leq h^{-p} \max_{t\neq s}\{[E(\varepsilon_s^{\xi})]^{1/\xi}\{E[(M(X_t)K_{ts})^{\eta}]\}^{1/\eta} = h^{-p}O(h^{p/\eta}) \\
&= O(h^{-(\eta-1)p/\eta}) \quad \text{for some } 1<\eta<2.
\end{aligned}$$

Hence $(\hat{\gamma}-\gamma_0)'A_{2n}(\hat{\gamma}-\gamma_0) = O_p((nh^{(\eta-1)p/\eta})^{-1}) = o_p((nh^{p/2})^{-1})$. Summarizing the above, we have shown that $I_{2n} = o_p((nh^{p/2})^{-1})$.

Proof of (iv): $I_{3n} = O_p(n^{-1})$.

By the mean value theorem and (C2) (i), we have $|g(x,\hat{\gamma})-g(x,\gamma_0)| \leq cM(x)\|\hat{\gamma}-\gamma_0\|$. Hence $I_{3n} \leq (c^2/n^2h^p)\sum\sum_{t\neq s}M(X_t)M(X_s)\, K_{ts}\|\hat{\gamma}-\gamma_0\|^2 \equiv S_n\|\hat{\gamma}-\gamma_0\|^2 = O_p(n^{-1})$ because $E\,|\,S_n| = O(1)$ by (C2) (iii) and $\hat{\gamma}-\gamma_0 = O_p(n^{-1/2})$. ∎

Consistency of this test against H_1 is proved in Fan and Li (1996b).

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APPENDIX A: PROOF OF THEOREM 2.1

Throughout this appendix, c is a generic positive constant. We first give a lemma that will be used to prove Theorem 2.1.

LEMMA A.0 *Let $\xi_1, \ldots, \xi_n$ be random vectors taking values in R^p satisfying an absolute regularity condition with coefficient β_m. Let $h(y,z)$ be a Borel measurable function, η be an $\mathcal{M}_1^k$-measurable random vector, and ζ be an $\mathcal{M}_{k+m}^{\infty}$-measurable random vector such that for some $\delta > 0$, $M = max\{E[|h(\eta,\zeta)|^{1+\delta}], \int\int |h(y,z)|^{1+\delta} Q(dy)R(dz)\}$ exists, where Q and R are probability distributions of η and ζ respectively. Further, let $g(y) = E[h(y,\zeta)]$. Then*

$$E|E\{h(\eta,\zeta)|\mathcal{M}_1^k\} - g(\eta)| \leq 4M^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}.$$

Proof The proof is similar to the proof of Lemma 2 in Takahata and Yoshihara (1987) and is thus omited here.

We recall that $U_n = \sum\sum_{1 \leq s < t \leq n} H(Z_t, Z_s)$. Let $\mathcal{F}_\alpha = \mathcal{M}_1^{a_\alpha - m+1}$ $(\alpha = 1, \ldots, k)$. Define, for $\alpha = 1, \ldots, k$, $T_\alpha = \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha - m+1} H(Z_t, Z_s)$ $(\alpha = 1, \ldots, k)$, and $V_n = \sum_{\alpha=1}^k T_\alpha$. We will show that V_n and U_n have the same asymptotic distribution. We first prove some lemmas.

LEMMA A.1 $\bar{s}_n^2 = [(n^2/2) + o(n^2)]\sigma_n^2$, *where* $\bar{s}_n^2 = E(V_n^2)$.

Proof It follows from the definition that $\bar{s}_n^2 = E(V_n^2) = E[(\sum_{\alpha=1}^k T_\alpha)^2] = \sum_{\alpha=1}^k E(T_\alpha^2) + 2\sum_{1\leq\alpha<\alpha'\leq k} E(T_\alpha T_{\alpha'})$. Define $Y_\alpha(x) = \sum_{s=1}^{a_\alpha - m+1} H(x, Z_s)$ $(\alpha = 1, \ldots, k)$. Then $T_\alpha = \sum_{t=a_\alpha}^{b_\alpha} Y_\alpha(Z_t)$ and

$$\begin{aligned} E(T_\alpha^2) &= E\left\{\sum_{t=a_\alpha}^{b_\alpha} Y_\alpha^2(Z_t)\right\} + 2E\left\{\sum_{a_\alpha \leq t < t' \leq b_\alpha} Y_\alpha(Z_t) Y_\alpha(Z_{t'})\right\} \\ &= I_\alpha + 2II_\alpha, \quad \text{(say)}. \end{aligned}$$

Note that

$$I_\alpha = E\left\{\sum_{t=a_\alpha}^{b_\alpha} Y_\alpha^2(Z_t)\right\} = \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha - m+1} \sum_{s'=1}^{a_\alpha - m+1} E[H(Z_t, Z_s) H(Z_t, Z_{s'})].$$

Below, we discuss three possible cases: $s=s'$; $1 \le |s-s'| \le m$; and $|s-s'|>m$.

(a) If $s=s'$, then by definition and Lemma 1 in Yoshihara (1976),

$$\begin{aligned} I_{\alpha(a)} &= \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha-m+1} E[H^2(\tilde{Z}_t,\tilde{Z}_s)] + \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha-m+1} \{E[H^2(Z_t,Z_s)] - E[H^2(\tilde{Z}_t,\tilde{Z}_s)]\} \\ &= r(a_\alpha - m+1)\sigma_n^2 + O(ra_\alpha M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}); \end{aligned}$$

(b) If $1 \le |s-s'| \le m$, then $I_{\alpha(b)} \le cra_\alpha m\gamma_n$;
(c) If $|s-s'|>m$, then $I_{\alpha(c)} \le 0 + cra_\alpha^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$ by Lemma 1 in Yoshihara (1976).

Hence, using $r\sum_{\alpha=1}^k (a_\alpha - m+1) = (n^2/2) + o(n^2)$, $r\sum_{\alpha=1}^k 1 \le n$ and $a_\alpha \le n$, we get $\sum_{\alpha=1}^k I_\alpha \equiv \sum_{\alpha=1}^k (I_{\alpha(a)} + I_{\alpha(b)} + I_{\alpha(c)}) = [(n^2/2)+o(n^2)]$ $\sigma_n^2 + O(n^2 m\gamma_n) + O(n^3 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}) = [(n^2/2)+o(n^2)]\sigma_n^2$, by Assumptions (A1) and (A3).

Next observe that

$$II_\alpha = \sum \sum_{a_\alpha \le t < t' \le b_\alpha} \sum \sum_{1 \le s,s' \le a_\alpha - m+1} E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})].$$

One can easily show that

(a) If $\max(|t-t'|,|s-s'|) > m$, then $|E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})]| \le 0 + cM_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$;
(b) If $\max(|t-t'|,|s-s'|) \le m$, then $|E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})]| \le \gamma_{n11}$.

Thus, $|II_\alpha| \le ra_\alpha m^2\gamma_n + cr^2 a_\alpha^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$, and $\sum_{\alpha=1}^k |II_\alpha| \le n^2$ $m^2\gamma_n + crn^3 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} = o(n^2\sigma_n^2)$, given (A1) and (A3). Finally we consider the case of $\alpha < \alpha'$: By Lemma 1 in Yoshihara (1976), we have $E(T_\alpha T_{\alpha'}) = \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha-m+1} \sum_{t'=a_{\alpha'}}^{b_{\alpha'}} \sum_{s'=1}^{a_{\alpha'}-m+1} E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})] \le 0$ $+cr^2 a_\alpha a_{\alpha'} M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$, because $t' - \max(t,s,s') \ge m$. Hence, $\sum\sum_{1 \le \alpha < \alpha' \le k} E(T_\alpha T_{\alpha'}) \le cn^4 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} = o(n^2\sigma_n^2)$, given (A3). Summarizing the above, we have shown that $\bar{s}_n^2 = (n^2/2)\sigma_n^2 + o(n^2\sigma_n^2)$.

LEMMA A.2

(i) $(1/\bar{s}_n)\sum_{\alpha=1}^k E(T_\alpha|F_\alpha) \xrightarrow{p} 0$, *and*
(ii) $(1/\bar{s}_n^2)\sum_{\alpha=1}^k \{E(T_\alpha^2|F_\alpha) - [E(T_\alpha|F_\alpha)]^2\} \xrightarrow{p} 1$.

Proof of (i) By Lemma A.0, we have $\sum_{\alpha=1}^{k} E|E(T_\alpha|F_\alpha)| \leq \sum_{\alpha=1}^{k} \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha-m+1} E|E\{H(Z_t,Z_s)|F_\alpha\}| \leq 0+cr\sum_{\alpha=1}^{k} a_\alpha M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} \leq cn^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} = o(n\sigma_n) = o(\bar{s}_n)$ by (A3). Hence $\sum_{\alpha=1}^{k} E(T_\alpha|F_\alpha) = o_p(\bar{s}_n)$.

Proof of (ii) From the proof of (i) above, we know that $(\bar{s}_n^2)^{-1} \sum_{\alpha=1}^{k} [E(T_\alpha|F_\alpha)]^2 \leq c(\bar{s}_n^2)^{-1} r^2 \sum_{\alpha=1}^{k} a_\alpha^2 M_n^{2/(1+\delta)}\beta_m^{2\delta/(1+\delta)} = o(1)$, given (A3).

It remains to prove: $(\bar{s}_n^2)^{-1} \sum_{\alpha=1}^{k} E(T_\alpha^2|F_\alpha) \xrightarrow{p} 1$. Let $A_n = \sum_{\alpha=1}^{k} E(T_\alpha^2|F_\alpha) = \sum_{\alpha=1}^{k} a_{n\alpha}$. Then

$$\begin{aligned} a_{n\alpha} &= E\left\{ \left[\sum_{t=a_\alpha}^{b_\alpha} Y_\alpha(Z_t) \right]^2 \middle| F_\alpha \right\} \\ &= \sum_{t=a_\alpha}^{b_\alpha} E[Y_\alpha^2(Z_t)|F_\alpha] + 2 \sum_{a_\alpha \leq t < t' \leq b_\alpha} E[Y_\alpha(Z_t)Y_\alpha(Z_{t'})|F_\alpha] \\ &= I_{F\alpha} + 2II_{F\alpha}, \quad \text{(say)}. \end{aligned}$$

Define $G(x,y) = E[H(Z_t,x)\,H(Z_t,y)]$. Since $I_{F\alpha} = \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha-m+1} \sum_{s'=1}^{a_\alpha-m+1} E[H(Z_t,Z_s)H(Z_t,Z_{s'})|F_\alpha]$, we get by using Lemma A.0:

$$\begin{aligned} &E\left| I_{F\alpha} - \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha-m+1} \sum_{s'=1}^{a_\alpha-m+1} G(Z_s,Z_{s'}) \right| \\ &\quad \leq \sum_{t=a_\alpha}^{b_\alpha} \sum_{s=1}^{a_\alpha-m+1} \sum_{s'=1}^{a_\alpha-m+1} E|E[H(Z_t,Z_s)H(Z_t,Z_{s'})|F_\alpha] - G(Z_s,Z_{s'})| \\ &\quad = O(ra_\alpha^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{\alpha=1}^{k} I_{F\alpha} &= r\sum_{\alpha=1}^{k} \sum_{s=1}^{a_\alpha-m+1} \sum_{s'=1}^{a_\alpha-m+1} G(Z_s,Z_{s'}) + O_p\left(r\sum_{\alpha=1}^{k} a_\alpha^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} \right) \\ &= r\sum_{\alpha=1}^{k} \sum_{s=1}^{a_\alpha-m+1} \sum_{s'=1}^{a_\alpha-m+1} G(Z_s,Z_{s'}) + O_p\left(n^3 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\right) \\ &= r\sum_{\alpha=1}^{k} \sum_{s=1}^{a_\alpha-m+1} \sum_{s'=1}^{a_\alpha-m+1} G(Z_s,Z_{s'}) + o_p(\bar{s}_n^2) \equiv \bar{A}_n + o_p(\bar{s}_n^2), \text{ given (A3)}. \end{aligned}$$

Next we show that $(\bar{A}_n/\bar{s}_n^2) = 1 + o_p(1)$. For this purpose, we note that

$$\begin{aligned}\bar{A}_n &= r\sum_{\alpha=1}^{k}\sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1} G(Z_s, Z_{s'})\\ &= r\sum_{\alpha=1}^{k}\sum_{s=1}^{a_\alpha-m+1} G(Z_s, Z_s) + r\sum_{\alpha=1}^{k}\sum\sum_{1\le s\neq s'\le a_\alpha-m+1} G(Z_s, Z_{s'})\\ &\equiv \bar{A}_{1n} + \bar{A}_{2n}.\end{aligned}$$

Below we will show: $\bar{A}_{1n} = \bar{s}_n^2(1 + o_p(1))$ and $\bar{A}_{2n} = o_p(\bar{s}_n^2)$. In fact we will prove a stronger result of $\bar{A}_{2n} = o_p(\bar{s}_n^2/m)$.

Proof of $\bar{A}_{1n} = \bar{s}_n^2(1 + o_p(1))$

By the proof of Lemma A.1, we know that $\bar{s}_n^2 = \sum_{\alpha=1}^{k}\sum_{t=a_\alpha}^{b_\alpha}\sum_{s=1}^{a_\alpha-m+1} E[H^2(\tilde{Z}_t, \tilde{Z}_s)] + o(\bar{s}_n^2) \equiv E(\bar{A}_{1n}) + o(\bar{s}_n^2)$. Hence, it suffices to show that $\mathrm{var}(\bar{A}_{1n}) = o(\bar{s}_n^4)$. It follows from the definition that

$$\begin{aligned}&\mathrm{var}(\bar{A}_{1n})\\ &= r^2\sum_{\alpha=1}^{k}\sum_{\alpha'=1}^{k}\sum_{s=1}^{a_\alpha-m+1} E\{[G(Z_s, Z_s) - EG(Z_s, Z_s)]^2\}\\ &\quad + r^2\sum_{\alpha=1}^{k}\sum_{\alpha'=1}^{k}\sum_{s=1}^{a_\alpha-m+1}\sum_{1\le|s-s'|\le m; s'=1}^{a_{\alpha'}-m+1} E\{[G(Z_s, Z_s) - EG(Z_s, Z_s)]\\ &\qquad\qquad [G(Z_{s'}, Z_{s'}) - EG(Z_{s'}, Z_{s'})]\}\\ &\quad + r^2\sum_{\alpha=1}^{k}\sum_{\alpha'=1}^{k}\sum_{s=1}^{a_\alpha-m+1}\sum_{|s-s'|>m; s'=1}^{a_{\alpha'}-m+1} E\{[G(Z_s, Z_s) - EG(Z_s, Z_s)]\\ &\qquad\qquad [G(Z_{s'}, Z_{s'}) - EG(Z_{s'}, Z_{s'})]\}\\ &\le n^3\sigma_g^2 + mn^3\sigma_G^2 + n^4 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} = o(n^4\sigma_n^4) = o(\bar{s}_n^4),\end{aligned}$$

by (A2) (iii) and (A3).

Proof of $\bar{A}_{2n} = o_p(\bar{s}_n^2/m)$

Write $\bar{A}_{2n} = r\sum_{\alpha=1}^{k}\sum\sum_{1\le s\neq s'\le a_\alpha-m+1} G(Z_s, Z_{s'}) \equiv r\sum_{\alpha=1}^{k}\bar{a}_{n\alpha}$. Then, for $\alpha < \alpha'$,

$$
\begin{aligned}
&E[\bar{a}_{n\alpha}\bar{a}_{n\alpha'}] \\
&= \sum \sum_{1\le s\ne s'\le a_\alpha-m+1} \sum \sum_{1\le s''\ne s'''\le a_\alpha-m+1} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})] \\
&\quad + \sum \sum_{1\le s\ne s'\le a_\alpha-m+1} \sum \sum_{a_\alpha-m+2\le s''\ne s'''\le a_{\alpha'}-m+1} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})] \\
&= 2\sum \sum_{1\le s\ne s'\le a_\alpha-m+1} E[G^2(Z_s,Z_{s'})] \\
&\quad + 2\sum\sum \sum_{1\le s\ne s'\ne s''\le a_\alpha-m+1} E[G(Z_s,Z_{s'})G(Z_s,Z_{s''})] \\
&\quad + \sum\sum\sum \sum_{1\le s\ne s'\ne s''\ne s'''\le a_\alpha-m+1} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})] \\
&\quad + \sum\sum \sum_{1\le s\ne s'\le a_\alpha-m+1,} \sum_{a_\alpha-m+2\le s''\ne s'''\le a_{\alpha'}-m+1} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})] \\
&\le 2a_\alpha^2\mu_{nG2} + (2a_\alpha^2 m\gamma_{nG11} + ca_\alpha^3 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}) + D_{1n} + D_{2n},
\end{aligned}
$$

where

$$
\begin{aligned}
D_{1n} &= \sum\sum\sum \sum_{1\le s\ne s'\ne s'''\le a_\alpha-m+1} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})], \\
D_{2n} &= \sum\sum\sum \sum_{1\le s\ne s'\le a_\alpha-m+1, a_\alpha-m+2\le s''\ne s'''\le a_{\alpha'}-m+1} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})].
\end{aligned}
$$

Because of symmetry, we only need to consider the case where $s < s' < s'' < s'''$ for D_{1n}:

(a) If $\max(s'-s, s'''-s'') > m$, then by Lemma 1 in Yoshihara (1976),

$$
E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})] \le 0 + cM_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)};
$$

(b) If $\max(s'-s, s'''-s'') \le m$, $E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})] \le \gamma_{nG11}$. Hence, $D_{1n} \le m^2a_\alpha^2\gamma_{nG11} + ca_\alpha^4 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$.

We now analyze D_{2n}:

$$
\begin{aligned}
&D_{2n} \\
&= \sum\sum\sum \sum_{1\le s\ne s'\le a_\alpha-m+1, a_\alpha+1\le s''\ne s'''\le a_\alpha-m+1} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})] \\
&\quad + \sum\sum\sum \sum_{1\le s\ne s'\le a_\alpha-m+1, a_\alpha-m+2\le s''\ne s'''\le a_\alpha} E[G(Z_s,Z_{s'})G(Z_{s''},Z_{s'''})]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big[\sum \sum_{1\leq s\neq s'\leq a_\alpha-m+1} EG(Z_s,Z_{s'})\Big]\Big[\sum \sum_{a_\alpha+1\leq s''\neq s'''\leq a_{\alpha'}-m+1} EG(Z_{s''},Z_{s'''})\Big] \\
&\quad + ca_\alpha^2 a_{\alpha'}^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} + m^2 a_\alpha^2 \gamma_{nG11} \\
&\leq [ma_\alpha \mu_{nG2}^{1/2} + ca_\alpha^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}][ma_{\alpha'}\mu_{nG2}^{1/2} + ca_{\alpha'}^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}] \\
&\quad + ca_\alpha^2 a_{\alpha'}^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} + m^2 a_\alpha a_{\alpha'}\gamma_{nG11} \\
&\leq 2[m^2 a_{\alpha'}^2 \mu_{nG2} + ca_{\alpha'}^4 M_n^{2/(1+\delta)}\beta_m^{2\delta/(1+\delta)}] + ca_\alpha^2 a_{\alpha'}^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} \\
&\quad + m^2 a_\alpha a_{\alpha'}\gamma_{nG11}.
\end{aligned}
$$

Therefore, for $\alpha < \alpha'$, we have

$$
\begin{aligned}
E[\bar{a}_{n\alpha}\bar{a}_{n\alpha'}] &\leq m^2 a_\alpha a_{\alpha'}\gamma_{nG11} + 2m^2 a_{\alpha'}^2\mu_{nG2} \\
&\quad + ca_\alpha^2 a_{\alpha'}^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} + ca_{\alpha'}^4 M_n^{2/(1+\delta)}\beta_m^{2\delta/(1+\delta)}.
\end{aligned}
$$

It is easy to see that $E[\bar{a}_{n\alpha}^2] \leq E[\bar{a}_{n\alpha}\bar{a}_{n\alpha'}]$ $(\alpha < \alpha')$. Hence the above bounds is also valid for $\alpha = \alpha'$. Substituting the above expression into $E[\bar{A}_{2n}^2]$ yields

$$
\begin{aligned}
E[\bar{A}_{2n}^2] &= r^2 \sum_{\alpha=1}^{k} E[\bar{a}_{n\alpha}^2] + 2r^2 \sum\sum_{1\leq\alpha<\alpha'\leq k} E[\bar{a}_{n\alpha}\bar{a}_{n\alpha'}] \\
&\leq c\{n^4 m^2\gamma_{nG11} + n^4 m^2 \mu_{nG2} + n^6 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} \\
&\quad + n^6 M_n^{2/(1+\delta)}\beta_m^{2\delta/(1+\delta)}\} = o(n^4\sigma_n^4/m^2) = o(\bar{s}_n^4/m^2),
\end{aligned}
$$

by Assumptions (A2) and (A3). This finishes the proof of $(\bar{A}_n/\bar{s}_n^2) = 1 + o_p(1)$.

In remains to show $\sum_{\alpha=1}^{k} II_{F\alpha} = o_p(\bar{s}_n^2)$. Define $\bar{G}_{(t-t')}(Z_s,Z_{s'}) = \int H(z_t,Z_s)H(z_{t'},Z_{s'})dF_{t,t'}(z_t,z_{t'})$. Then we obtain by Lemma A.0,

$$
\begin{aligned}
&E\Big| II_{F\alpha} - \sum\sum_{a_\alpha\leq t<t'\leq b_\alpha} \sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1} \bar{G}_{(t-t')}(Z_s,Z_{s'})\Big| \\
&\quad \leq \sum\sum_{a_\alpha\leq t<t'\leq b_\alpha} \sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1} E|E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})|F_\alpha] \\
&\quad\quad - \int H(z_t,Z_s)H(z_{t'},Z_{s'})dF_{t,t'}(z_t,z_{t'})| \leq Cr^2 a_\alpha^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}.
\end{aligned}
$$

Hence,

$$\begin{aligned}\sum_{\alpha=1}^{k} II_{F\alpha} &= \sum_{\alpha=1}^{k}\sum\sum_{a_\alpha\le t<t'\le b_\alpha}\sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1}\bar{G}_{(t-t')}(Z_s,Z_{s'})\\ &\quad + O_p\left(rn^3M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\right)\\ &= \sum_{\alpha=1}^{k}\sum\sum_{a_\alpha\le t<t'\le b_\alpha}\sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1}\bar{G}_{(t-t')}(Z_s,Z_{s'}) + o_p(\bar{s}_n^2)\\ &\equiv \bar{B}_n + o_p(\bar{s}_n^2),\end{aligned}$$

by (A3), where

$$\begin{aligned}\bar{B}_n &= \sum_{\alpha=1}^{k}\sum\sum_{|t'-t|\le m, a_\alpha\le t<t'\le b_\alpha}\sum_{s=1}^{a_\alpha-m+1}\bar{G}_{(t-t')}(Z_s,Z_s)\\ &\quad + \sum_{\alpha=1}^{k}\sum\sum_{|t'-t|\le m, a_\alpha\le t<t'\le b_\alpha}\sum_{1\le s'\neq s\le a_\alpha-m+1}\bar{G}_{(t-t')}(Z_s,Z_{s'})\\ &\quad + \sum_{\alpha=1}^{k}\sum\sum_{|t'-t|>m, a_\alpha\le t<t'\le b_\alpha}\sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1}\bar{G}_{(t-t')}(Z_s,Z_{s'})\\ &\equiv \bar{B}_{1n} + \bar{B}_{2n} + \bar{B}_{3n}.\end{aligned}$$

First $E|\bar{B}_{1n}| \le \sum_{\alpha=1}^{k} rma_\alpha\gamma_n \le n^2m\gamma_n = o(n^2\sigma_n^2) = o(\bar{s}_n^2)$ by (A1).

Next, consider $\bar{B}_{2n}$: Obviously $\bar{B}_{2n}$ has an order no larger than $m\bar{A}_{2n}$. Hence by the same proof of $\bar{A}_{2n} = o_p(\bar{s}_n^2/m)$, we get $\bar{B}_{2n} = o_p(\bar{s}_n^2)$.

Finally, $\bar{B}_{3n}$ satisfies

$$\begin{aligned}E|\bar{B}_{3n}| &\le \sum_{\alpha=1}^{k}\sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1}E\left|\sum\sum_{|t'-t|>m, a_\alpha\le t<t'\le b_\alpha}\bar{G}_{(t-t')}(Z_s,Z_{s'})\right|\\ &= \sum_{\alpha=1}^{k}\sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1}\\ &\quad \int\left|\sum\sum_{|t'-t|>m, a_\alpha\le t<t'\le b_\alpha}\right.\\ &\qquad \left.\int[H(z_t,z_s)H(z_{t'},z_{s'})]dF_{t,t'}(z_t,z_{t'})\right|dF_{s,s'}(z_s,z_{s'})\\ &\le \sum_{\alpha=1}^{k}\sum_{s=1}^{a_\alpha-m+1}\sum_{s'=1}^{a_\alpha-m+1}\end{aligned}$$

$$\int\left|\sum\sum_{|t'-t|>m,a_\alpha\leq t<t'\leq b_\alpha}(0+cM_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)})\right|dF_{s,s'}(z_s,z_{s'})$$
$$=O\left(\sum_{\alpha=1}^{k}a_\alpha^2r^2M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\right)=O(rn^3M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)})$$
$$=o(n^2\sigma_n^2)=o(\bar{s}_n^2),$$

by (A3). This completes the proof of Lemma A.2.

LEMMA A.3 $(1/\bar{s}_n^4)\sum_{\alpha=1}^{k}E(T_\alpha^4)=o(1)$.

Proof Let Q_α be the distribution function of $(Z_{a_\alpha},\dots,Z_{b_\alpha})$. By C_r-inequality, we have $\max\{E|T_\alpha^4|^{1+\delta},E\int|T_\alpha^4|^{1+\delta}dQ_\alpha\}\leq(rn)^{4(1+\delta)}M_n$. Hence, by Lemma A.0, we obtain

$$E(T_\alpha^4)\leq E\left\{\int\left[\sum_{t=a_\alpha}^{b_\alpha}Y_\alpha(z_t)\right]^4dQ_\alpha\right\}+cn^4r^4M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$$
$$\equiv L_\alpha+cn^4r^4M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)},$$

where

$$L_\alpha=\int\sum_{t=a_\alpha}^{b_\alpha}E[Y_\alpha^4(z_t)]dQ_\alpha+\int\sum\sum_{a_\alpha\leq t\neq t'\leq b_\alpha}E[Y_\alpha^2(z_t)Y_\alpha^2(z_{t'})]dQ_\alpha$$
$$+\int\sum\sum_{a_\alpha\leq t\neq t'\leq b_\alpha}E[Y_\alpha^3(z_t)Y_\alpha(z_{t'})]dQ_\alpha$$
$$+\int\sum\sum\sum_{a_\alpha\leq t_1\neq t_2\neq t_3\leq b_\alpha}E[Y_\alpha^2(z_{t_1})Y_\alpha(z_{t_1})Y_\alpha(z_{t_3})]dQ_\alpha$$
$$+\int\sum\sum\sum\sum_{a_\alpha\leq t_1\neq t_2\neq t_3\neq t_4\leq b_\alpha}E[Y_\alpha(z_{t_1})Y_\alpha(z_{t_2})Y_\alpha(z_{t_3})Y_\alpha(z_{t_4})]dQ_\alpha$$
$$=L_{\alpha1}+L_{\alpha2}+L_{\alpha3}+L_{\alpha_4}+L_{\alpha5}.$$

Let $\sum_{(k)}$ be the summation over all $s_1,\dots,s_k$ such that $1\leq s_1,\dots,s_k\leq a_\alpha-m+1$ and $s_i\neq s_j\,(i\neq j)$. $L_{\alpha1}$ contains terms of $\sum_{(1)},\dots,\sum_{(4)}$. The bounds for these cases are given in (A.1) to (A.5) below.

$$\sum_{(1)}\int E[H^4(z,Z_s)]dF(z)\leq a_\alpha\mu_{n4}. \tag{A.1}$$

By Lemma 1 in Yoshihara (1976), we have

$$\sum_{(2)}\left|\int E[H^3(z,Z_{s_1})H(z,Z_{s_2})]dF(z)\right|$$
$$=\left\{\sum_{(2),|s_1-s_2|\le m}+\sum_{(2),|s_1-s_2|>m}\right\}\left|\int E\,[H^3(z,Z_{s_1})H\,(z,Z_{s_2})\,]dF(z)\right|$$
$$\le a_\alpha m\nu_n + ca_\alpha^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}. \tag{A.2}$$

Obviously,

$$\sum_{(2)}\int E[H^2(x,Z_{s_1})H^2(x,Z_{s_2})]dF(x)$$
$$=\left\{\sum_{1\le|s_1-s_2|\le m}+\sum_{|s_1-s_2|>m}\right\}\int E[H^2(x,Z_{s_1})H^2(x,Z_{s_2})dF(x)]$$
$$\le a_\alpha m\nu_n + a_\alpha^2\{\gamma_n + cM_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\}. \tag{A.3}$$

Next we evaluate the order of $\sum_{(3)}$: we consider three possible cases: (a) $|s_2-s_3|>2m$; (b) $|s_2-s_3|\le 2m$ and $\min\{|s_1-s_2|,|s_1-s_3|\}>m$; (c) $|s_2-s_3|\le 2m$ and $\min\{|s_1-s_2|,s_1-s_3|\}\le m$. First note that if $|t_3-t_2|>2m$, then $\max(|t_2-t_1|,|t_3-t_1|)>m$. Then by Lemma 1 Yoshihara (1976), we have

$$\sum_{(3)}\left|\int E\{H^2(z,Z_{s_1})H(z,Z_{s_2})H(z,Z_{s_3})\}dF(z)\right.$$
$$=\left\{\sum_{(a)}+\sum_{(b)}+\sum_{(c)}\right\}\left|\int E\{H^2(z,Z_{s_1})H^2(x,Z_{s_2})H\,(z,Z_{s_3})\}dF(z)\right.$$
$$\le c\{a_\alpha^3 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} + a_\alpha^2 m\gamma_n + a_\alpha m^2\nu_n\}. \tag{A.4}$$

Now we consider the $\sum_{(4)}$ term, we ditinguish three sub-cases:

(a) If $s_1<s_2<s_3<s_4$ and $\max(s_2-s_1,s_4-s_3)>m$, then by Lemma 1 in Yoshihara (1976), $|\int E\{\Pi_{i=1}^4 H(z,Z_{s_i})\}dF(z)|\le c(M_n+M_n)^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$;
(b) If $s_1<s_2<s_3<s_4$, $\max(s_2-s_1,s_4-s_3)\le m$ and $s_3-s_2>m$, then $|\int E\{\Pi_{i=1}^4 H(z,Z_{s_i})\}dF(z)|\le\gamma_n + c(M_n+M_n)^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$;
(c) If $s_1<s_2<s_3<s_4$, $\max(s_2-s_1,s_4-s_3)\le m$ and $s_3-s_2\le m$, then $|E\{\Pi_{i=1}^4 H(z,Z_{s_i})\}dF(z)|\le\nu_n$.

Hence, we have

$$\sum_{(4)}\left|\int E\{\Pi_{i=1}^{4}H(z,Z_{s_i})\}dF(z)\right| \leq ca_\alpha^4 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}+a_\alpha^2 m^2\gamma_n+a_\alpha m^3\nu_n. \tag{A.5}$$

Now we are ready to show that $\sum_{\alpha=1}^{k} L_{\alpha j} = o(\bar{s}_n^4)$ for $j = 1,\ldots,5$.

Proof of $\sum_{\alpha=1}^{k} L_{\alpha 1} = o(\bar{s}_n^4)$

Define

$$G_{\alpha,z} = \int E[Y_\alpha^4(z)]dF(z) \sim \sum_{(1)}+\sum_{(2)}+\sum_{(2)}+\sum_{(3)}+\sum_{(4)}. \tag{A.6}$$

Note that although $G_{\alpha,z} = G_\alpha$ we will write the argument z for reasons that will be apparent later. By (A.1)–(A.5), we have

$$G_{\alpha,z} \leq ca_\alpha\{\mu_{n4} + a_\alpha m^2\gamma_n + m^3\nu_n + a_\alpha^3 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\}. \tag{A.7}$$

Assumptions (A1), (A3), Eq. (A.7) and the fact that $\bar{s}_n^4 = O(n^4\sigma_n^4)$ lead to

$$\begin{aligned}\sum_{\alpha=1}^{k} L_{\alpha 1} &= \sum_{\alpha=1}^{k}\sum_{t=a_\alpha}^{b_\alpha}\int E[Y_\alpha^4(z_t)]dF(z_t)\\ &\leq c\{n^2\mu_{n4} + n^3m^2\gamma_n + n^2m^3\nu_n\\ &\qquad + n^5 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\} = o(\bar{s}_n^4).\end{aligned} \tag{A.8}$$

Proof of $\sum_{\alpha=1}^{k} L_{\alpha 2} = o(\bar{s}_n^4)$

By Schwartz's inequality (used twice) and (A.7), we have

$$\begin{aligned}\sum_{\alpha=1}^{k} L_{\alpha 2} &= \sum_{\alpha=1}^{k}\sum_{a_\alpha\leq t\neq t'\leq b_\alpha}\int E[Y_\alpha^2(z_t)Y_\alpha^2(z_{t'})]dF(z_t,z_{t'})\\ &\leq \sum_{\alpha=1}^{k}\sum_{a_\alpha\leq t\neq t'\leq b_\alpha}\{G_{\alpha,z_t}G_{\alpha,z_{t'}}\}^{1/2}\\ &\leq cr^2\sum_{\alpha=1}^{k} ca_\alpha\{\mu_{n4}+a_\alpha m^2\gamma_n+m^3\nu_n+a_\alpha^3 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\}\\ &= cr\{n^2\mu_{n4}+n^3m^2\gamma_n+n^2m^3\nu_n+n^5M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\}\\ &= o(n^4\sigma_n^4) = o(\bar{s}_n^4), \quad \text{given (A1) and (A3)}.\end{aligned} \tag{A.9}$$

Proof of $\sum_{\alpha=1}^{k} L_{\alpha 3} = o(\bar{s}_n^4)$

The proof is similar to the proof of $\sum_{\alpha=1}^{k} L_{\alpha 2} = o(\bar{s}_n^4)$.

Proof of $\sum_{\alpha=1}^{k} L_{\alpha 4} = o(\bar{s}_n^4)$

Note that if $|t_3 - t_2| > 2m$, then $\max(|t_2 - t_1|, |t_3 - t_1|) > m$. Hence by Lemma 1 in Yoshihara (1976), we have in this case,

$$B(t_1, t_2, t_3) \equiv \left| E \int Y_\alpha^2(z_{t_1}) Y_\alpha(z_{t_2}) Y_\alpha(z_{t_3}) dF(z_{t_1}, z_{t_2}, z_{t_3}) \right| \\ \leq 0 + c a_\alpha^4 (M_n + M_n)^{1/(1+\delta)} \beta_m^{\delta/(1+\delta)}.$$

Hölder's inequality and (A.7) lead to

$$\begin{aligned}
\sum_{\alpha=1}^{k} L_{\alpha 4} &\leq \sum_{\alpha=1}^{k} \left\{ \sum_{a_\alpha \leq t_1, t_2, t_3 \leq b_\alpha; |t_3 - t_2| > 2m} + \sum_{a_\alpha \leq t_1, t_2, t_3 \leq b_\alpha; |t_3 - t_2| \leq 2m} \right\} B(t_1, t_2, t_3) \\
&\leq \sum_{\alpha=1}^{k} \left\{ \sum \sum \sum_{a_\alpha \leq t_1, t_2, t_3 \leq b_\alpha; |t_3 - t_2| \leq 2m} [G_{\alpha, t_1}]^{1/2} [G_{\alpha, t_2}]^{1/4} [G_{\alpha, t_3}]^{1/4} \right. \\
&\qquad \left. + \sum \sum \sum_{a_\alpha \leq t_1, t_2, t_3 \leq b_\alpha; |t_3 - t_2| > 2m} c a_\alpha^4 (M_n + M_n)^{1/(1+\delta)} \beta_m^{\delta/(1+\delta)} \right\} \\
&\leq c r^2 m \sum_{\alpha=1}^{k} c a_\alpha \{ \mu_{n4} + m^2 a_\alpha \gamma_n + m^3 \nu_n + a_\alpha^3 M_n^{1/(1+\delta)} \beta_m^{\delta/(1+\delta)} \} \\
&\qquad + c r^3 \sum_{\alpha=1}^{k} a_\alpha^4 M_n^{1/(1+\delta)} \beta_m^{\delta/(1+\delta)} \\
&= rm\, O(n^2 \mu_{n4} + m^2 n^3 \gamma_n + n^2 m^3 \nu_n + n^5 M_n^{1/(1+\delta)} \beta_m^{\delta/(1+\delta)}) \\
&\qquad + O(r^2 n^5 M_n^{1/(1+\delta)} \beta_m^{\delta/(1+\delta)}) \\
&= o(n^4 \sigma_n^4) c = o(\bar{s}_n^4), \quad \text{given (A1) and (A3).}
\end{aligned} \tag{A.10}$$

Proof of $\sum_{\alpha=1}^{k} L_{\alpha 5} = o(\bar{s}_n^4)$

Since $s_1 \neq s_2 \neq s_3 \neq s_4$, it suffices to show that

$$G_n \equiv \sum_{\alpha=1}^{k} \sum \sum \sum \sum_{a_\alpha \leq t_1 < t_2 < t_3 < t_4 \leq b_\alpha} \int E\left\{ \prod_{l=1}^{4} Y(Z_{t_l}) \right\} dQ_\alpha = o(n^4 \sigma_n^4). \tag{A.11}$$

By Lemma 1 in Yoshihara (1976) and Hölder's inequality, we have

$$\begin{aligned}
G_n &\le \sum_{\alpha=1}^{k}\Big[\sum_{(1)'}+\sum_{(2)'}\Big]\Big|\int E\Big\{\prod_{l=1}^{4}Y(z_{t_l})\Big\}dQ_\alpha\Big| \\
&\le \sum_{\alpha=1}^{k}\sum_{(1)'}\Big|\int E\Big\{\prod_{l=1}^{4}Y(z_{t_l})\Big\}dQ_\alpha\Big| + \sum_{(2)'}\prod_{l=1}^{4}\{G_{\alpha,z_{t_l}}\}^{1/4} \\
&\le cr^4\sum_{\alpha=1}^{k}a_\alpha^4 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} \\
&\quad + cr^2m^2\sum_{\alpha=1}^{k}a_\alpha\{\mu_{n4}+a_\alpha m^2\gamma_n+m^3\nu_n+a_\alpha^3M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\} \\
&\le cr^3n^5M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} \\
&\quad + crm^2\{n^2\mu_{n4}+n^3m^2\gamma_n+n^2m^3\nu_n+n^5M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\} \\
&= o(n^4\sigma_n^4) = o(\bar{s}_n^{\,4}) \quad \text{given (A1) and (A3)}
\end{aligned}$$

where $\sum_{(1)'}$ ($\sum_{(2)'}$) denotes the summation over all t_l $(l = 1, \ldots, 4)$ such that $a_\alpha \le t_1 < t_2 < t_3 < t_4 \le b_\alpha$, and $\max(t_2 - t_1, t_4 - t_3) > m$ ($\max(t_2 - t_1, t_4 - t_3) \le m$).

Thus, we have proved that

$$\begin{aligned}
\sum_{\alpha=1}^{k}E(T_\alpha^4) &= \sum_{\alpha=1}^{k}\{L_{\alpha1}+L_{\alpha2}+L_{\alpha3}+L_{\alpha4}+L_{\alpha5}\} \\
&\quad + cr^3n^5M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} \\
&= o(n^4\sigma_n^4) = o(\bar{s}_n^4),
\end{aligned}$$

finishing the proof of Lemma A.3.

LEMMA A.4 $U_n - V_n = o_p(\bar{s}_n)$.

Proof We recall that $U_n = V_n + B_n + Q_n$, where $V_n = \sum_{\alpha=1}^{k}\sum_{t=a_\alpha}^{b_\alpha}\sum_{s=1}^{a_\alpha-m+1}H(Z_t, Z_s)$, $B_n = \sum_{\alpha=1}^{k}\sum_{t=a_\alpha}^{b_\alpha}\sum_{a_\alpha-m+2\le s<t}H(Z_t,Z_s)$, and $Q_n = \sum_{\alpha=1}^{k}\sum_{t=b_\alpha+1}^{a_{\alpha+1}-1}\sum_{s<t}H(Z_t,Z_s)$. It easy to see that $Q_n = O(m/r)$ $O_p(V_n+B_n) = o_p(V_n+B_n)$. Also be Lemma A.1, we know that $V_n = O_p(\bar{s}_n)$. Hence in order to prove the lemma, it suffices to show that

$B_n = o_p(\bar{s}_n)$ or $E[(B_n^2)] = o(\bar{s}_n^2)$. To this end, we note that

$$
\begin{aligned}
&E[(B_n^2)]\\
&= E\left\{\left[\sum_{\alpha=1}^{k}\sum_{t=a_\alpha}^{b_\alpha}\sum_{a_\alpha-m+2\le s<t} H(Z_t,Z_s)\right]^2\right\}\\
&= \sum_{\alpha=1}^{k}\sum_{t=a_\alpha}^{b_\alpha}\sum_{t'=a_\alpha}^{b_\alpha}\sum_{a_\alpha-m+2\le s<t}\sum_{a_\alpha-m+2\le s'<t'} E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})]\\
&\quad +2\sum_{1\le\alpha<\alpha'\le k}\sum_{t=a_\alpha}^{b_\alpha}\sum_{t'=a_{\alpha'}}^{b'_\alpha}\sum_{a_\alpha-m+2\le s<t}\sum_{a_{\alpha'}-m+2\le s'<t'} E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})]\\
&\equiv R_{1n}+R_{2n}.
\end{aligned}
$$

For R_{1n}, we consider three possible cases for the summation indices t, s, t', s': (a) the four summation indices take two different values; (b) they take three different values; and (c) all four indices are different.

Case (a) We have $R_{1na} \le \sum_{\alpha=1}^{k} r(r+m)\{\sigma_n^2 + CM_n \sum_{\alpha=1}^{k}\sum_{t=a_\alpha}^{b_\alpha} \sum_{s\ge a_\alpha-m+1}^{t-1} \beta_{t-s}^{\delta/(1+\delta)}\} = O(nr\sigma_n^2) + O(n) = o(n^2\sigma_n^2)$ given (A1);

Case (b) $R_{1n(b)} \le \sum_{\alpha=1}^{k} r(r+m)^2\gamma_n \le n(r+m)^2\gamma_n = o(n^2\sigma_n^2)$ given (A1);

Case (c) $R_{1n(c)} \le \sum_{\alpha=1}^{k} r^2(r+m)^2\gamma_n \le nr(r+m)^2\gamma_n = o(n^2\sigma_n^2)$ given (A1). Hence $R_{1n} = R_{1n(a)} + R_{1n(b)} + R_{1n(c)} = o(\bar{s}_n^2)$.

For R_{2n}, we consider two cases: (i) $\max\{t'-s', t-s\} > m$; (ii) $\max\{t'-s', t-s\} \le m$.

Case (i) If $t'-s' > m$, then $t' - \max(s', t, s) > m$; If $t-s > m$, then $\min(t', s', t) - s > m$.

In either case, $E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})] \le 0 + cM^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}$. Hence, we have by (A3)

$$
\begin{aligned}
R_{2n(i)} &\le c\sum_{\alpha=1}^{k}\sum_{\alpha'=1}^{k} r^2(r+m)^2 M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)}\\
&\le cn^2r^2M_n^{1/(1+\delta)}\beta_m^{\delta/(1+\delta)} = o(n^2\sigma_n^2).
\end{aligned}
$$

Case (ii) Using $E[H(Z_t,Z_s)H(Z_{t'},Z_{s'})] \le \gamma_n$, we obtain $R_{2n(ii)} \le c \sum_{\alpha=1}^{k}\sum_{\alpha'=1}^{k} r^2m^2\gamma_n \le n^2m^2\gamma_n = o(n^2\sigma_n^2)$ by (A1). Hence, $R_{2n} = o(\bar{s}_n^2)$ and $E(B_n^2) = R_{1n} + R_{2n} = o(\bar{s}_n^2)$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemmas A.2 and A.3, we know that $(1/\bar{s}_n)\sum_{\alpha=1}^{k} E(T_\alpha|F_\alpha) \xrightarrow{p} 0$, $(1/\bar{s}_n^2)\sum_{\alpha=1}^{k}\{E(T_\alpha^2|F_\alpha) - [E(T_\alpha|F_\alpha)]^2\} \xrightarrow{p} 1$, and $(1/\bar{s}_n^4)\sum_{\alpha=1}^{k} E(T_\alpha^4) \xrightarrow{p} 0$. Hence, by Theorem 2.2 in Dvoretzky (1972), we get $(1/\bar{s}_n)V_n \to N(0,1)$ in distribution.

Moreover, Lemmas A.1 and A.4 imply, $\bar{s}_n^2 = [(n^2/2) + o(n^2)]\sigma_n^2$ and $U_n - V_n = o_p(\bar{s}_n)$. Hence, $\sqrt{2}U_n/(n\sigma_n) \to N(0,1)$ in distribution.